INVARIANCE OF O-MINIMAL COHOMOLOGY WITH DEFINABLY COMPACT SUPPORTS

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Abstract. In this paper we find general criteria to ensure that, in an arbitrary o-minimal structure, the o-minimal cohomology without supports and with definably compact supports of a definable space with coefficients in a sheaf is invariant in elementary extensions and in o-minimal expansions. We apply our criteria and obtain new invariance results for the o-minimal cohomology of: (a) definable spaces in o-minimal expansions of ordered groups and (b) definably compact definable groups in arbitrary o-minimal structures. We also prove the o-minimal analogues of Wilder's finiteness theorem in these two contexts.

1. Introduction

In this paper we find a general criteria Theorem 4.6 (resp. Theorem 4.7) to ensure that, in an arbitrary o-minimal structure $\mathbb{M}$, the o-minimal cohomology without supports $H^*(X;F)$ (resp. with definably compact supports $H^*_c(X;F)$) of a definable space X with coefficients in a sheaf F on the o-minimal site on X is invariant in elementary extensions and in o-minimal expansions of $\mathbb{M}$. Our criteria apply to the following cases:

Theorem 1.1. *Suppose that $\mathbb{M}$ is an o-minimal expansion of an ordered group. Let X be a definably normal definable space which is definably completable by a definably normal definable space. Let F be a sheaf on the o-minimal site on X. If $\mathbb{S}$ is an elementary extension of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$), then we have*

$$H^*_c(X;F) \simeq H^*_c(X(\mathbb{S});F(\mathbb{S})).$$

In particular, by Theorem 1.1 we have:

Corollary 1.2. *Suppose that $\mathbb{M}$ is an o-minimal expansion of an ordered group. Let X be a definably normal, definably compact, definable space. Let F be a sheaf on the o-minimal site on X. If $\mathbb{S}$ is an elementary extension of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$), then we have*

$$H^*(X;F) \simeq H^*(X(\mathbb{S});F(\mathbb{S})).$$

Note that if $\mathbb{M}$ is an o-minimal expansion of an ordered group, then Hausdorff, definably compact definable manifolds are definably normal (see [2, Lemma 10.4] - which is proved there is in o-minimal expansions of ordered fields but the same proof works in our case as it only uses the existence of a continuous definable distance

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and definable choice). So for definably compact definable manifolds, the hypothesis of Corollary 1.2 can be weakened.

Since in o-minimal expansions of ordered fields regular definable spaces are affine and so definably normal ([10, Chapter 10, (1.8) and Chapter 6, (3.8)]) and are definably completable by affine definable spaces ([10, Chapter 10 (2.5)]), by Theorem 1.1 we have:

Corollary 1.3. *Suppose that* $\mathbb{M}$ *is an o-minimal expansion of an ordered field. Let* X *be a regular definable space. Let* F *be a sheaf on the o-minimal site on* X*. If* $\mathbb{S}$ *is an elementary extension of* $\mathbb{M}$ *(resp. an o-minimal expansion of* $\mathbb{M}$*), then we have*

$$H_c^*(X;F) \simeq H_c^*(X(\mathbb{S});F(\mathbb{S})).$$

Observe that if $\mathbb{M}$ is an o-minimal expansion of an ordered group, then Hausdorff definable manifolds are regular (see [14, Proposition 2.2] - which is proved there is in o-minimal expansions of fields but the same proofs works in our case as it only uses the existence a of continuous definable distance and definable choice). So for definable manifolds, the hypothesis of Corollary 1.3 can be weakened.

Another useful case where our main criteria applies is the following:

Theorem 1.4. *Suppose that* $\mathbb{M}$ *is an arbitrary o-minimal structure. Let* G *be a definably compact definable group. Let* F *be a sheaf on the o-minimal site on* G*. If* $\mathbb{S}$ *is an elementary extension of* $\mathbb{M}$ *(resp. an o-minimal expansion of* $\mathbb{M}$*), then we have*

$$H^*(G;F) \simeq H^*(G(\mathbb{S});F(\mathbb{S})).$$

Our Corollary 1.3 is a generalization to the o-minimal case of a similar result is real closed fields due to Delfs ([9, Theorem 6.10]). Delfs proof of this result in real closed fields is based on the semi-algebraic triangulation theorem and the same method applies as well in o-minimal expansions of fields using the o-minimal triangulation theorem ([10, Chapter 8, (2.9)]) instead. However, of course, this method does not generalize to our other invariance results. Our method also allows us to prove an analogue of Corollary 1.3 without supports (Corollary 5.6) which generalizes the comparison results from [18] for cohomology with constant coefficients.

On the other hand, our Corollary 1.2 is a generalization of a similar result for closed and bounded definable sets (i.e. affine definably compact definable spaces) in o-minimal expansions of ordered groups. See [1] and also [3] for a similar result in o-minimal expansions of fields. Note however that in general, in arbitrary o-minimal expansions of ordered groups, definably normal definable spaces (even definably compact ones) need not be affine as in o-minimal expansions of fields. See [19] and [10, Chapter 10, (1.8)].

Theorem 1.4 is an important step towards the computation of the o-minimal cohomology of definably compact definable groups in arbitrary o-minimal structures which are expected to be similar to the o-minimal cohomology of definably compact definable groups definable in o-minimal expansions of fields ([12]).

We point out also that the results of this paper play a crucial role in the development of the formalism of the Grothendieck six operations on o-minimal sheaves. See [16]. Indeed, we develop such theory in o-minimal structures with definable

choice and in full a subcategory **A** of the category of definable spaces (in such o-minimal structures) whose set of objects is:

- closed under taking definable subspaces of objects of **A**,
- closed under taking cartesian products of objects of **A**,

and, is such that:

(A1) every object of **A** is definably normal;
(A2) every object of **A** is definably completable in **A**;
(A3) for every object X of **A**, for every model $\mathbb{S}$ of the first-orer theory of $\mathbb{M}$ and for every sheaf F on the o-minimal site on X we have an isomorphism

$$H^*_c(X; F) \simeq H^*_c(X(\mathbb{S}); F(\mathbb{S})).$$

In this paper we find three important examples of such subcategories **A**: (i) the full subcategory of regular definable spaces in o-minimal expansions of real closed fields; (ii) the full subcategory of definably normal spaces in o-minimal expansions of ordered groups which have definably normal completions; (iii) the full subcategory of definable subspaces of cartesian products of a given definably compact definable group in an arbitrary o-minimal structure.

In this paper we also prove the following o-minimal Wilder's finiteness theorems:

Theorem 1.5. *Suppose that $\mathbb{M}$ is an o-minimal expansion of an ordered group. Suppose that L is a finitely generated module over a noetherian ring. Let X be a definably compact, definably normal definable space whose definable charts (X_i, ϕ_i) are such that each $\phi_i(X_i)$ is a bounded definable set. Then, for each p, $H^p(X; L_X)$ is finitely generated.*

This result is a generalization of a similar result for closed and bounded definable sets (i.e. affine definably compact definable spaces) in o-minimal expansions of ordered groups. See [1] and also [3] for a similar result in o-minimal expansions of fields.

Theorem 1.6. *Suppose that $\mathbb{M}$ is an arbitrary o-minimal structure. Suppose that L is a finitely generated module over a noetherian ring. Let G be a definably compact definable group. Then, for each p, $H^p(G; L_G)$ is finitely generated.*

In the paper [16] we use this theorem together with Theorem 1.4 to show that in arbitrary o-minimal structures the o-minimal cohomology $H^*(G; k_G)$ with coefficients in a field k of a definably connected, definably compact definable group G is a connected, bounded Hopf algebra of finite type.

2. On definable normality

Here we introduce the category of definable spaces (already present in [10, Chapter 10]) and make preliminary observations about definable normality that will be useful later.

2.1. **Definable normality.**

Here we make some basic observations about the definable analogues of the topological separation axioms.

Let $\mathbb{M}$ be an arbitrary o-minimal structure. First recall that in M we have the order topology generated by open definable intervals and in M^k we have the

product topology whose basis are the cartesian products of k open intervals. Thus every definable set $X \subseteq M^k$ has the induced topology and we say that a definable subset $Z \subseteq X$ is open (resp. closed) if it is open (resp. closed) with the induced topology. Similarly, we can talk about continuous definable maps $f : X \to Y$ between definable sets.

Since we do not want to restrict our work to the affine definable setting, we introduce the notion of definable spaces ([10]).

Definition 2.1. A *definable space* is a triple $(X, (X_i, \theta_i)_{i \leq k})$ where:

- $X = \bigcup_{i \leq k} X_i$;
- each $\theta_i : X_i \to M^{n_i}$ is an injection such that $\theta_i(X_i)$ is a definable subset of M^{n_i} with the induced topology;
- for all i, j, $\theta_i(X_i \cap X_j)$ is an open definable subset of $\theta_i(X_i)$ and the transition maps $\theta_{ij} : \theta_i(X_i \cap X_j) \to \theta_j(X_i \cap X_j) : x \mapsto \theta_j(\theta_i^{-1}(x))$ are definable homeomorphisms.

We call the (X_i, θ_i)'s the *definable charts of* X and define the dimension of X by $\dim X = \max\{\dim \theta_i(X_i) : i = 1, \ldots, k\}$. If all the $\theta_i(X_i)$'s are open definable subsets of some M^n, we say that X is a *definable manifold of dimension* n.

A definable space X has a topology such that each X_i is open and the θ_i's are homeomorphisms: a subset U of X is an open in the basis for this topology if and only if for each i, $\theta_i(U \cap X_i)$ is an open definable subset of $\theta_i(X_i)$.

A map $f : X \to Y$ between definable spaces with definable charts $(X_i, \theta_i)_{i \leq k}$ and $(Y_j, \delta_j)_{j \leq l}$ respectively is *a definable map* if:

- for each i and every j with $f(X_i) \cap Y_j \neq \emptyset$, $\delta_j \circ f \circ \theta_i^{-1} : \theta_i(X_i) \to \delta_j(Y_j)$ is a definable map between definable sets.

We say that a definable space is *affine* if it is definably homeomorphic to a definable set with the induced topology.

The construction above defines the *category of definable spaces with definable continuous maps* which we denote by Def. All topological notions on definable spaces are relative to the topology above. Note however, that often we will have to replace topological notions on definable spaces by their definable analogue.

We say that a subset A of a definable space X is definable if and only if for each i, $\theta_i(A \cap X_i)$ is a definable subset of $\theta_i(X_i)$. A definable subset A of a definable space X is naturally a definable space and its topology is the induced topology, thus we also call them *definable subspaces*. From the corresponding result for definable sets ([13, Proposition 2.1]), we obtain:

Remark 2.2. *Every definable subset of a definable space is a finite union of definable subsets of the form $U \cap F$ where U (resp. F) is an open (resp. closed) definable subset of X.*

In non-standard o-minimal structures definable sets and so definable spaces are usually totally disconnected and never connected. Thus we say that a definable space X is *definably connected* if it is not the disjoint union of two open and closed definable subsets.

Since the basis for the topology in M^k are the cartesian products of k open intervals and the total order in M is assumed to be dense without endpoints, definable

sets and so affine definable spaces are Hausdorff (in particular T_1 and T_0). Hence, we have:

Remark 2.3. *Every definably space is T_1, i.e. points are closed.*

However, definable spaces are not in general Hausdorff:

Example 2.4 (Non Hausdorff definable space)**.** Let $a, b, c, d \in M$ be such that $c < b < a < d$. Let X be the definable space with definable charts $(X_i, \theta_i)_{i=1,2}$ given by: $X_1 = (\{\langle x, y\rangle \in (c, d) \times (c, d) : x = y\} \setminus \{\langle b, b\rangle\}) \cup \{\langle b, a\rangle\} \subseteq M^2$, $X_2 = \{\langle x, y\rangle \in (c, d) \times (c, d) : x = y\} \subseteq M^2$ and $\theta_i = \pi_{|X_i}$ where $\pi : M^2 \to M$ is the projection onto the first coordinate. Then any open definable neighborhood in X of the point $\langle b, a\rangle$ intersects any open definable neighborhood in X of the point $\langle b, b\rangle$.

A topological space X is *regular* if one the following equivalent conditions holds:

(1) for every $a \in X$ and $S \subseteq X$ closed such that $a \notin S$, there are open disjoint subsets U and V of X such that $a \in U$ and $S \subseteq V$;
(2) for every $a \in X$ and $W \subseteq X$ open such that $a \in W$, there is V open subset of X such that $a \in V$ and $\overline{V} \subseteq W$;

Since the basis for the topology in M^k are the cartesian products of k open intervals and the total order in M is assumed to be dense without endpoints, definable sets and so, affine definable spaces, are regular. However, definable spaces are not in general regular and in fact, Hausdorff definable spaces are not necessarily regular:

Example 2.5 (Hausdorff non regular definable space)**.** Let $a, b, c, d \in M$ be such that $c < b < a < d$. Let X be the definable space with definable charts $(X_i, \theta_i)_{i=1,2}$ given by: $X_1 = \{\langle x, y\rangle \in (c, d)\times(c, d) : x < y\}\cup\{\langle a, a\rangle\} \subseteq M^2$, $X_2 = (b, a)\times(b, a) \subseteq M^2$ and $\theta_i = \mathrm{id}_{M^2|X_i}$. Let $C = \{\langle x, y\rangle \in X_2 : x = y\}$. Then X is Hausdorff. On the other hand, $\langle a, a\rangle \notin C$, $\langle a, a\rangle$ is closed in X and C is closed in X (as it is disjoint from X_1 and closed in X_2). Also, any open definable neighborhood in X of the point $\langle a, a\rangle$ intersects any open definable neighborhood in X of C.

This is a modification of Robson's example of a Hausdorff definable space an o-minimal expansion of an ordered field which is not regular ([10, page 159]).

A definable space X is *definably normal* if one of the following equivalent conditions holds:

(1) for every disjoint closed definable subsets Z_1 and Z_2 of X there are disjoint open definable subsets U_1 and U_2 of X such that $Z_i \subseteq U_i$ for $i = 1, 2$.
(2) for every $S \subseteq X$ closed definable and $W \subseteq X$ open definable such that $S \subseteq W$, there is an open definable subsets U of X such that $S \subseteq U$ and $\overline{U} \subseteq W$.

Example 2.6 (Regular non definably normal definable space)**.** Assume that $\mathbb{M} = (M, <)$ is a dense linearly ordered set with no end points. Let $a, b, c, d \in M$ be such that $c < b < a < d$ and let $X = (c, d) \times (c, d) \setminus \{\langle a, b\rangle\}$. Since X is affine it is regular. Note also that the only open definable subsets of X are the intersections with X of definable subsets of M^2 which are finite unions of non empty finite intersections $W_1 \cap \cdots \cap W_k$ where each W_i is either an open box in M^2, $\{\langle x, y\rangle \in M^2 : x < y\}$ or $\{\langle x, y\rangle \in M^2 : y < x\}$.

Let $C = \{\langle x, y\rangle \in X : x = a\}$ and let $D = \{\langle x, y\rangle \in X : y = b\}$. Then C and D are closed disjoint definable subsets of X. However, by the description of the open definable subset of X, there are no open disjoint definable subsets U and V of X such that $C \subseteq U$ and $D \subseteq V$.

As usual we have (compare with [10, Chapter 6, (3.6)]):

Fact 2.7 (The shrinking lemma)**.** *Suppose that X is a definably normal definable space. If $\{U_i : i = 1, \dots, n\}$ is a covering of X by open definable subsets, then there are definable open subsets V_i and definable closed subsets C_i of X ($1 \leq i \leq n$) with $V_i \subseteq C_i \subseteq U_i$ and $X = \cup\{V_i : i = 1, \dots, n\}$.*

A definable space X is *completely definably normal* if one of the following equivalent conditions holds:

1. every definable subset Z of X is a definably normal definable subspace.
2. every open definable subset U of X is a definably normal definable subspace.
3. for every closed definable subsets Z_1 and Z_2 of X, if $Z_0 = Z_1 \cap Z_2$, then there are open definable subsets V_1 and V_2 of X such that:
   (i) $Z_i \setminus V_i = Z_0$, $i = 1, 2$.
   (ii) $V_1 \cap V_2 = \emptyset$.
   (iii) $\overline{V_1} \cap \overline{V_2} \subseteq Z_0$.
4. for every definable subsets S_1 and S_2 of X, if $S_1 \cap \overline{S_2} = \overline{S_1} \cap S_2 = \emptyset$, then there are disjoint open definable subsets U_1 and U_2 of X such that $S_i \subseteq U_i$ for $i = 1, 2$.

Example 2.8 (Definably normal non completely definably normal space)**.** The closure $\overline{X} = [c, d] \times [c, d]$ of the non definably normal definable space $X = (c, d) \times (c, d) \setminus \{\langle a, b\rangle\}$ of Example 2.6 is definably normal.

By definition or Remark 2.3 we have:

Remark 2.9. *For definable spaces, completely definably normal implies definably normal, definably normal implies regular and regular implies Hausdorff.*

2.2. **Normality in elementary extensions and in o-minimal expansions.** Here we make some observations about the behavior of normality when going to elementary extensions or o-minimal expansions.

Let $\mathbb{S}$ be a model of the first-order theory of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$). As it is well known $\mathbb{S}$ determines a functor from the category of ($\mathbb{M}$-)definable sets and ($\mathbb{M}$-)definable maps to the category of $\mathbb{S}$-definable sets and $\mathbb{S}$-definable maps. This functor extends to a functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ sending a definable space X to the $\mathbb{S}$-definable space $X(\mathbb{S})$ and sending a continuous definable map $f : X \to Y$ to the continuous $\mathbb{S}$-definable map $f^{\mathbb{S}} : X(\mathbb{S}) \to Y(\mathbb{S})$.

Remark 2.10. *If $\mathbb{S}$ is model of the first-order theory of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$), then the following hold:*

1. *The functor* $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ *is a monomorphism from the boolean algebra of definable subsets of a definable space X to the boolean algebra of $\mathbb{S}$-definable subsets of $X(\mathbb{S})$ and it commutes with:*
   - *the interior and closure operations;*

- *the image and inverse image under (continuous) definable maps.*

(2) *A definable subset B of a definable space X is closed (resp. open) if and only if the $\mathbb{S}$-definable subset $B(\mathbb{S})$ of the $\mathbb{S}$-definable space X is closed (resp. open).*

(3) *A definable space X is definably connected if and only if the $\mathbb{S}$-definable space $X(\mathbb{S})$ is $\mathbb{S}$-definably connected.*

Suppose that $\mathbb{S}$ is model of the first-order theory of $\mathbb{M}$. The property Hausdorff is first-order and thus preserved under the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$. On the other hand, "regular" and "definably normal" are not first-order but instead they are second-order properties since they involve quantification over (closed) definable subsets. In particular, in general, we do not have invariance of regular and definably normal under the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ (unless we have uniformly regular and uniformly definably normal).

Suppose that $\mathbb{S}$ is an o-minimal expansion of $\mathbb{M}$. The property Hausdorff is preserved under the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$ (by Remark 2.10 and since $X = X(\mathbb{S})$.) On the other hand, "regular" and "definably normal" are in general, not invariant under the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$.

In the next subsection we will find conditions under which definable normality is invariant under the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$. See Proposition 2.20.

2.3. **Normality in o-minimal spectra.** Here we introduce the category of o-minimal spectra of definable spaces which is given by the tilde functor and make some observations about normality in these spaces.

Definition 2.11. The *o-minimal spectrum* $\widetilde{X}$ of a definable space X is, as in the affine case ([6], [8] and [28]), the set of ultrafilters of definable subsets of X (also called in model theory, types concentrated on X) equipped with the topology generated by the open subsets of the form $\widetilde{U}$, where U is an open definable subset of X.

The *o-minimal spectrum* $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ *of a (continuous) definable map* $f : X \to Y$ between definable spaces is the (continuous) map such that given an ultrafilter $\alpha \in \widetilde{X}$, $f(\alpha)$ is the ultrafilter in $\widetilde{Y}$ determined by the collection $\{A : f^{-1}(A) \in \alpha\}$.

The *category of o-minimal spectra of definable spaces together o-minimal spectra of continuous definable maps*, denoted $\widetilde{\mathrm{Def}}$, is the category such that:

- the objects are of the form $\widetilde{X}$ where X is an object of Def;
- the morphisms are of the form $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ where $f : X \to Y$ is a morphism of Def.

The tilde operation is a functor $\mathrm{Def} \to \widetilde{\mathrm{Def}}$ and is an isomorphism of categories.

Fact 2.12. *The o-minimal spectrum $\widetilde{X}$ of a definable space X is T_0, quasi-compact and a spectral topological space, i.e., it has a basis of quasi-compact open subsets, closed under taking finite intersections and each irreducible closed subset is the closure of a unique point.*

All topological notions on o-minimal spectra of definable spaces are relative to this topology, which we wil call the *spectral topology.* The spectral topology is not

T_1 (unless X is finite), namely not every point is closed. In particular it is not Hausdorff (unless X is finite).

We now recall some results from [13] about this tilde functor. Note that these results were stated in [13] in the category of definable sets but are true in the category of definable spaces with exactly the same proofs. In fact most of them hold also in real algebraic spaces ([4], [8]) and more generally in spectral topological space ([7]).

First recall that if X is a definable space then: (i) a subset $A \subseteq \widetilde{X}$ is *constructible* if it is a finite boolean combination of the basic open subsets $\widetilde{U} \subseteq \widetilde{X}$ (equivalently, by Remark 2.2, if and only is it is of the form $\widetilde{C}$ for some definable subset C of X); (ii) for $\alpha, \beta \in \widetilde{X}$, we say that β is a *specialization* of α or α is a *generalization* of β, denoted $\alpha \rightsquigarrow \beta$, if and only if β is contained in the closure of $\{\alpha\}$ in $\widetilde{X}$. The notion of specialization is valid in any spectral space and defines a partial order on the set of points.

Remark 2.13. *For a definable space X the following hold:*

1. *The tilde functor is an isomorphism between the boolean algebra of definable subsets of X and the boolean algebra of constructible subsets of $\widetilde{X}$ and it commutes with:*
   - *the interior and closure operations;*
   - *the image and inverse image under (continuous) definable maps.*
2. *A constructible subset of $\widetilde{X}$ is closed (resp. open) if and only if it is stable under specialization (resp. generalization).*
3. *X is definably connected if and only if $\widetilde{X}$ is connected.*

Remark 2.14. *Let X be a definable space and for $\alpha \in \widetilde{X}$ define*

$$\dim(\alpha) = \min\{\dim A : A \in \alpha\}.$$

Then any chain of specializations of α has size at most $\dim(\alpha)$*. In particular, there exists at least one closed specialization of α*

It is easy to see that $\widetilde{X}$ is not regular unless X is finite. Regarding normality we have:

Fact 2.15. *For a definable space X the following are equivalent:*

1. *X is definably normal;*
2. *$\widetilde{X}$ is normal;*
3. *any closed point of $\widetilde{X}$ has a basis of closed neighborhoods in $\widetilde{X}$;*
4. *any two distinct closed points of $\widetilde{X}$ can be separated by disjoint open subsets of $\widetilde{X}$;*
5. *any point of $\widetilde{X}$ has a unique closed specialization.*

Also we have the following stronger shrinking lemma:

Fact 2.16 (The shrinking lemma)**.** *Suppose that X is a definably normal definable space. If $\{U_i : i = 1, \ldots, n\}$ is a covering of $\widetilde{X}$ by open subsets, then there are constructible open subsets V_i and constructible closed subsets K_i of $\widetilde{X}$ ($1 \leq i \leq n$) with $V_i \subseteq K_i \subseteq U_i$ and $\widetilde{X} = \cup\{V_i : i = 1, \ldots, n\}$.*

Regarding complete normality we have:

Fact 2.17. *For a definable space X the following are equivalent:*

(1) *X is completely definably normal;*
(2) *the specializations of any point of $\widetilde{X}$ form a chain.*

A proof of the first implication in the result was given by A. Fornasiero [21] (in o-minimal expansions of ordered groups, but the proof works in general) using the characterization of completely definably normal given before. For completeness we include another shorter proof based on Fact 2.15.

We prove the result by induction on the maximal size of a chain of specializations of a point in the o-minimal spectra of a completely definably normal definable space. Such chains are finite by Remark 2.14. Let X be a completely definably normal definable space and $\alpha \in \widetilde{X}$. If the size of the maximal chain of specializations of α is one, then α is closed and the result holds. Otherwise, since X is definably normal, by Fact 2.15, let τ be the unique closed specialization of α. Let β and γ be two non closed specializations of α. It is enough to show that either $\beta \rightsquigarrow \gamma$ or $\gamma \rightsquigarrow \beta$. We have that $\widetilde{X} \setminus \{\tau\}$ is an open subset of $\widetilde{X}$ and hence there is an open constructible subset $\widetilde{W} \subseteq \widetilde{X} \setminus \{\tau\}$ such that $\alpha, \beta, \gamma \in \widetilde{W}$. Since W is also completely definably normal and the size of the maximal chain of specializations of α in $\widetilde{W}$ has dropped, by the inductive hypothesis, either $\beta \rightsquigarrow \gamma$ or $\gamma \rightsquigarrow \beta$ in $\widetilde{W}$. But then, clearly, the same thing holds in $\widetilde{X}$ as required.

For the other implication in the result, suppose that the specializations of any point of $\widetilde{X}$ form a chain. We will show that X is completely definably normal. Let $U \subseteq X$ be an open definable subset and $\alpha \in \widetilde{U}$. Then the specializations of α in $\widetilde{U}$ are also specializations of α in $\widetilde{X}$. Therefore, the specializations of α in $\widetilde{U}$ form a finite sub-chain of the finite chain of specializations of α in $\widetilde{X}$ (Remark 2.14). The minimal element of that sub-chain is the unique closed specialization of α in $\widetilde{U}$. By Fact 2.15, U is a definably normal space as required.

The following is often useful:

Fact 2.18. *Let X be a completely definably normal definable space. If $\alpha \in \widetilde{X}$, then there is an open definable subset U of X such that $\alpha \in \widetilde{U}$ and α is a closed point of $\widetilde{U}$.*

This is proved by induction of the size of the chain of specialization of α. If this size is one, α is closed in $\widetilde{X}$. Otherwise, let ρ be the unique closed specialization of α. Then $\widetilde{X} \setminus \{\rho\}$ is open and $\alpha \in \widetilde{X} \setminus \{\rho\}$. So there is an open definable subset V of X such that $\alpha \in \widetilde{V} \subseteq \widetilde{X} \setminus \{\rho\}$. We have that V is a completely definably normal definable space and the size of the chain of specialization of α in $\widetilde{V}$ has dropped (it is a sub-chain of the previous chain). By the induction hypothesis, there is an open definable subset U of V such that $\alpha \in \widetilde{U}$ and α is a closed point of $\widetilde{U}$. But U is also an open definable subset of X.

Definition 2.19. Let $X_1, \dots, X_l$ be definable spaces and let $\mathcal{P}$ be a property of l-tuples of definable spaces. We say that $\mathcal{P}$ is *affine on* $(X_1, \dots, X_l)$ if $\mathcal{P}$ holds on $(X_1, \dots, X_l)$ whenever it holds on every l-tuple $(C_1, \dots, C_l)$ of closed affine definable subspace $C_i \subseteq X_i$.

In this paper we shall use the following general strategy. To prove that a property $\mathcal{P}$ of definable spaces holds on a definably normal definable space X we: (a) use the shrinking lemma to prove that $\mathcal{P}$ is affine on X; (b) prove $\mathcal{P}$ for affine definable subspaces of X. Of course, in each case the proofs of (a) and (b) are specific to the given $\mathcal{P}$. Observe also that not every $\mathcal{P}$ is affine, e.g. the property saying I am affine is not affine since the semi-linear group which is not affine from [19] is definably compact and hence definably normal by [17, Corollary 2.3].

Proposition 2.20. *Let $\mathbb{S}$ be a model of the first-order theory of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$). Let X and Y be definably normal definable spaces. Then the following properties are affine on (X, Y):*

1. *$X \times Y$ is definably normal;*
2. *X completely definably normal;*
3. *$Y(\mathbb{S})$ is $\mathbb{S}$-definably normal.*

Proof. Let $(X_i, \phi_i)_{i \leq l_X}$ be the definable charts of X and let $(Y_j, \psi_j)_{j \leq l_Y}$ be the definable charts of Y. By the shrinking lemma, there are open definable subsets V_i $(1 \leq i \leq l_X)$ and closed definable subsets C_i $(1 \leq i \leq l_X)$ such that $V_i \subseteq C_i \subseteq X_i$ and $X = \cup\{C_i : i = 1, \ldots, l_X\}$. Similarly, there are open definable subsets W_j $(1 \leq j \leq l_Y)$ and closed definable subsets D_j $(1 \leq j \leq l_Y)$ such that $V_j \subseteq D_j \subseteq Y_j$ and $Y = \cup\{D_j : j = 1, \ldots, l_Y\}$.

Clearly we then have $X \times Y = \cup\{C_i \times D_j : i = 1, \ldots, l_X \text{ and } j = 1, \ldots, l_Y\}$ and $Y(\mathbb{S}) = \cup\{D_j(\mathbb{S}) : j = 1, \ldots, l_Y\}$.

Let $\alpha \in \widetilde{X \times Y}$ (resp. $\alpha \in \widetilde{X}$ and $\alpha \in \widetilde{Y(\mathbb{S})}$). Then there is (i, j) (resp. i and j) such that $\alpha \in \widetilde{C_i \times D_j}$ (resp. $\alpha \in \widetilde{C_i}$ and $\alpha \in \widetilde{D_j(\mathbb{S})}$). Since $\widetilde{C_i \times D_j}$ (resp. C_i and D_j) is closed constructible, the specializations of α in $\widetilde{X \times Y}$ (resp. in $\widetilde{X}$ and in $\widetilde{Y(\mathbb{S})}$) are all in $\widetilde{C_i \times D_j}$ (resp. in $\widetilde{C_i}$ and in $\widetilde{D_j(\mathbb{S})}$) by Remark 2.13 (2). Now $C_i \subseteq X$ and $D_j \subseteq Y$ are closed affine definable subspaces. Therefore, if $C_i \times D_j$ is definably normal (resp. C_i is completely definably normal and $D_j(\mathbb{S})$ is $\mathbb{S}$-definably normal), then by Fact 2.15 (resp. Fact 2.17 and Fact 2.15), α has a unique closed specialization in $\widetilde{C_i \times D_j}$ (resp. the specializations of α in $\widetilde{C_i}$ form a chain and α has a unique closed specialization in $\widetilde{D_j(\mathbb{S})}$.) Hence, by Fact 2.15 (resp. Fact 2.17 and Fact 2.15), $X \times Y$ is definably normal (resp. X is completely definably normal and $Y(\mathbb{S})$ is $\mathbb{S}$-definably normal). $\square$

3. On definable compactness

Here we will make some observations about definable compactness of definable spaces in arbitrary o-minimal structures. Most of our observations were already known in the affine case ([26]) or in the affine case in o-minimal expansions of ordered groups ([10, Chapter 6]).

Let X be a definable space and $C \subseteq X$ a definable subset. By a *definable curve in C* we mean a continuous definable map $\alpha : (a, b) \to C \subseteq X$, where $a < b$ are in $M \cup \{-\infty, +\infty\}$. We say that a definable curve $\alpha : (a, b) \to C \subseteq X$ in C is *completable in C* if both limits $\lim_{t \to a^+} \alpha(t)$ and $\lim_{t \to b^-} \alpha(t)$ exist in C.

The next useful lemma is a generalization of its affine version ([26, Theorem 2.3] and [10, Chapter 6, (1.5)]) to definable spaces.

Lemma 3.1 (Almost everywhere curve selection)**.** *Suppose that U is a definable space. Let $C \subseteq U$ be a definable subset which is not closed. Then there is a definable set $E \subseteq \overline{C} \setminus C$ such that $\dim E < \dim(\overline{C} \setminus C)$ and for every $x \in \overline{C} \setminus (C \cup E)$ there is a definable curve in C which has x as a limit point. In particular, a definable subset $B \subseteq U$ is closed if and only if every definable curve in B which is completable in U is completable in B.*

Proof. Consider the definable charts $(U_i, \phi_i)_{i=1}^k$ of U. We prove the result by induction on k. Suppose that $k = 1$, say (U, ϕ) is the only definable chart of U. Since ϕ is a definable homeomorphism, we can assume that $\phi = \mathrm{id}$ and say $U \subseteq M^n$ is an open definable subset. By [26, Theorem 2.3], there is a definable $F \subseteq \overline{C} \setminus C$, where the closure in taken in M^n, such that $\dim F < \dim(\overline{C} \setminus C)$ and for every $x \in \overline{C} \setminus (C \cup F)$ there is a definable curve in C which has x as a limit point. Let $E = F \cap U$. Then $E \subseteq \overline{C} \cap U \setminus C = (\overline{C} \setminus C) \cap U$, $\overline{C} \cap U$ is the closure of C in U and for every $x \in (\overline{C} \cap U) \setminus (C \cup E)$ there is a definable curve in C which has x as a limit point. Since E (resp. $\overline{C} \cap U \setminus C = (\overline{C} \setminus C) \cap U$) is relatively open in F (resp. $\overline{C} \setminus C$), we have $\dim E = \dim F$ and $\dim \overline{C} \cap U \setminus C = \dim \overline{C} \setminus C$ and so $\dim E < \dim(\overline{C} \cap U \setminus C)$ as required.

Suppose now that the result holds for definable spaces with less or equal than l definable charts and $k = l + 1$. Let $V = \bigcup_{i=1}^l U_i$, $W = U_k$ and $\phi = \phi_k$. Then V is an open definable subspace of U with l definable charts. On the other hand, $\overline{C} \cap V \setminus (C \cap V) = \overline{C} \cap V \setminus C = (\overline{C} \setminus C) \cap V$ and $\overline{C} \cap V$ is the closure of $C \cap V$ in V and similarly, $\overline{C} \cap W \setminus (C \cap W) = \overline{C} \cap W \setminus C = (\overline{C} \setminus C) \cap W$ and $\overline{C} \cap W$ is the closure of $C \cap W$ in W. By the inductive hypothesis, there is a definable set $E_V \subseteq \overline{C} \cap V \setminus (C \cap V)$ such that $\dim E_V < \dim(\overline{C} \cap V \setminus (C \cap V))$ and for every $x \in \overline{C} \cap V \setminus ((C \cap V) \cup E_V)$ there is a definable curve in $C \cap V$ which has x as a limit point. Similarly, there is a definable set $E_W \subseteq \overline{C} \cap W \setminus (C \cap W)$ such that $\dim E_W < \dim(\overline{C} \cap W \setminus (C \cap W))$ and for every $x \in \overline{C} \cap W \setminus ((C \cap W) \cup E_W)$ there is a definable curve in $C \cap W$ which has x as a limit point. Let $E = E_V \cup E_W$. Clearly, $\overline{C} \setminus C = [\overline{C} \cap V \setminus (C \cap V)] \cup [\overline{C} \cap W \setminus (C \cap W)]$ and so $E \subseteq \overline{C} \setminus C$ is a definable subset. Since $C = (C \cap V) \cup (C \cap W)$, for every $x \in \overline{C} \setminus (C \cup E)$ there is a definable curve in C which has x as a limit point. Since $\dim E = \max\{\dim E_V, \dim E_W\}$ and $\dim \overline{C} \setminus C = \max\{\dim(\overline{C} \cap V \setminus (C \cap V)), \dim(\overline{C} \cap W \setminus (C \cap W))\}$ and thus $\dim E < \dim(\overline{C} \setminus C)$ as required. $\square$

In nonstandard o-minimal structures closed and bounded definable sets are not compact. Thus we have to replace the notion of compactness by a suitable definable analogue.

Definition 3.2. Let X be a definable space and $C \subseteq X$ a definable subset. We say that C is *definably compact* if every definable curve in C is completable in C (see [26]).

With this definition we have that a definable set $X \subseteq M^n$ with its induced topology is definably compact if and only if it is closed and bounded in M^n ([26, Theorem 2.1]).

Note that it is not assumed that a definably compact definable space is necessarily Hausdorff, unlike in the topological case where a compact space is assumed to be Hausdorff.

By Almost curve selection (Lemma 3.1) we have:

Remark 3.3. *Suppose that K is a definable subset of a definable space X. If K is definably compact subset, then K is a closed definable subset.*

Proposition 3.4. *Let $\mathbb{S}$ be a model of the first-order theory of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$). Let X be a definably normal, definably compact definable space. Then $X(\mathbb{S})$ is an $\mathbb{S}$-definably compact $\mathbb{S}$-definable space.*

Proof. For definably normal definable spaces, by the shrinking lemma, definably compact is an affine property. Since "closed and bounded" is invariant in models and in o-minimal expansions of $\mathbb{M}$ the result follows. $\square$

We end the section with some observations about definable completions which will be useful later.

Definition 3.5. Let $\mathbf{B}$ be a subcategory of Def. We say that an object X of $\mathbf{B}$ is *definably completable in* $\mathbf{B}$ if there exists a definably compact space P in $\mathbf{B}$ together with a definable open immersion $\iota : X \hookrightarrow P$ in $\mathbf{B}$, i.e. $\iota(X)$ is open in P and $\iota : X \to \iota(X)$ is a definable homeomorphism, with $\iota(X)$ dense in P. Such $\iota : X \hookrightarrow P$ is called a *definable completion* of X in $\mathbf{B}$.

The following is easy:

Remark 3.6. Let $\mathbf{B}$ be a subcategory of Def. Then the following hold:

(1) If X is an object of $\mathbf{B}$ which is definably completable in $\mathbf{B}$ and $Z \subseteq X$ is a definable subspace of X which is an object of $\mathbf{B}$, then Z is definably completable in $\mathbf{B}$.

(2) If X and Y are objects of $\mathbf{B}$ which are definably completable in $\mathbf{B}$ and $X \times Y$ is an object of $\mathbf{B}$, then $X \times Y$ is definably completable in $\mathbf{B}$.

By [10, Chapter 10, (1.8), Chapter 10, (2.5)] we have:

Fact 3.7. *If $\mathbb{M}$ is an o-minimal expansion of a real closed field, then every regular definable space is definably completable by an affine definable space.*

However, in general definable completions do not exist:

Example 3.8. Suppose that there is an unbounded definable interval $X \subseteq M$ such that there is no definable bijection between X and a bounded definable interval. For example, this is the case if $\mathbb{M}$ is a semi-bounded o-minimal expansion of an ordered group ([11]). Then clearly, the definable space X has no definable completion.

Proposition 3.9. *Let X be a definable space with a definable completion $\iota : X \to P$ such that P is definably normal. Then X is a definably locally compact definable space, i.e. for every definably compact definable subset K of X and every open*

definable neighborhood U of K in X, there is a definably compact definable neighborhood C of K in U.

Proof. We have that $\iota(X)$ and $\iota(U)$ are open definable subsets of P and $\iota(K)$ is a definably compact definable subset of P disjoint from the closed definable subset $P \setminus \iota(U)$ of P. Now the result follows at once from the definable normality and definable compactness of P. □

4. O-minimal sheaves and invariance

In this section we recall basic facts about sheaves on topological spaces, about o-minimal sheaves and we prove the general invariance criteria for o-minimal sheaf cohomology without supports and for o-minimal sheaf cohomology with definably compact supports.

4.1. **Sheaves.** Let X be a topological space, let $\mathrm{Op}(X)$ be the category of open subsets of X (morphisms are given by the inclusions) and let A be a ring. We denote by $\mathrm{Mod}(A_X)$ the category of sheaves A-modules on X. We will call the objects of $\mathrm{Mod}(A_X)$ A-sheaves on X. Here we recall some notions and some useful facts about A-sheaves on topological spaces. These general notions and results apply also to A-sheaves on objects of $\widetilde{\mathrm{Def}}$. We refer to [5], [22], [23] and [24] for further details on these results and other results on A-sheaves on topological spaces that we will use later for A-sheaves on objects of $\widetilde{\mathrm{Def}}$.

An A-sheaf on X is a contravariant functor $F : \mathrm{Op}(X)^{\mathrm{op}} \to \mathrm{Mod}(A_X)$, $U \mapsto \Gamma(U; F)$ satisfying gluing conditions, which are described, for each $U \in \mathrm{Op}(X)$ and each covering $\mathcal{U} = \{U_i\}$ of U, by the exact sequence

$$0 \to \Gamma(U; F) \to \prod_{U_i \in \mathcal{U}} \Gamma(U_i; F) \rightrightarrows \prod_{U_j, U_k \in \mathcal{U}} \Gamma(U_j \cap U_k; F).$$

A fiber F_x of F on a point $x \in X$ is given by the limit $\varinjlim_{x \in U \in \mathrm{Op}(X)} \Gamma(U; F)$. A sequence $0 \to F \to G \to H \to 0$ is exact on $\mathrm{Mod}(A_X)$ if it is exact on fibers, i.e. if $0 \to F_x \to G_x \to H_x \to 0$ is exact for each $x \in X$.

Let $f : X \to Y$ be a continuous map. The functor $f_* : \mathrm{Mod}(A_X) \to \mathrm{Mod}(A_Y)$ of direct image is defined by $\Gamma(U; f_*F) = \Gamma(f^{-1}(U); F)$ for $F \in \mathrm{Mod}(A_X)$. The inverse image functor $f^{-1} : \mathrm{Mod}(A_Y) \to \mathrm{Mod}(A_X)$ is defined as follows: if $G \in \mathrm{Mod}(A_Y)$, then $f^{-1}G$ is the sheaf associated to the presheaf $U \mapsto \varinjlim_{U \subseteq f^{-1}(V)} \Gamma(V; F)$. The functor f^{-1} is left adjoint to the functor f_*, i.e. we have a functorial isomorphism $\mathrm{Hom}(f^{-1}G, F) \simeq \mathrm{Hom}(G, f_*F)$; the direct image functor f_* is left exact and commutes with small projective limits; the inverse image functor f^{-1} is exact and commutes with small inductive limits. When i_U is the inclusion of an open subset on X we have $i_U^{-1}F = F_{|U}$.

Let $i_Z : Z \to X$ be the inclusion of a locally closed subset Z of X. We recall the definition of the functor $i_{Z!}$ (*extension by zero*) such that for $F \in \mathrm{Mod}(A_Z)$, $i_{Z!}F$ is the unique A-sheaf in $\mathrm{Mod}(A_X)$ inducing F on Z and zero on $X \setminus Z$. First let U be an open subset of X and let $F \in \mathrm{Mod}(A_U)$. Then $i_{U!}F$ is the sheaf associated to the presheaf $V \mapsto \Gamma(V; i_{U!}F)$ which is $\Gamma(V; F)$ if $V \subseteq U$ and 0 otherwise. If S is a closed subset of X and $F \in \mathrm{Mod}(A_S)$, then $i_{S!}F = i_{S*}F$. Now let $Z = U \cap S$ be a locally closed subset of X, then one defines $i_{Z!} = i_{U!} \circ i_{S!} \simeq i_{S!} \circ i_{U!}$. If $f : X \to Y$ is a continuous map, Z a locally closed subset of Y,

$$\begin{tikzcd} f^{-1}(Z) \arrow[r, hook, "j"] \arrow[d, "f_|"] & X \arrow[d, "f"] \\ Z \arrow[r, hook, "i"] & Y \end{tikzcd} \tag{1}$$

a commutative diagram and $G \in \mathrm{Mod}(A_Z)$, then $f^{-1} \circ i_! G \simeq j_! \circ (f_|)^{-1} G$.

Let $F \in \mathrm{Mod}(A_X)$. One sets $F_Z = i_{Z!} \circ i_Z^{-1} F$. Thus F_Z is characterized by $F_{Z|Z} = F_{|Z}$ and $F_{Z|X\setminus Z} = 0$. It is an exact functor. If Z' is another locally closed subset of X, then $(F_Z)_{Z'} = F_{Z \cap Z'}$. Let L be an A-module. When $F = L_X$ is the constant sheaf on X of fiber L we just set L_Z instead of $(L_X)_Z$.

The functor $(\bullet)_Z$ admits a right adjoint, denoted by Γ_Z which is left exact. Let $V \in \mathrm{Op}(X)$. When $Z = U \in \mathrm{Op}(X)$ we have $\Gamma(V; \Gamma_U F) = \Gamma(U \cap V; F)$. When Z is closed $\Gamma(V; \Gamma_Z F) = \{s \in \Gamma(V; F) : \operatorname{supp} s \subseteq Z\}$.

Let Φ be a family of supports on X (i.e. a collection of closed subsets of X such that: (i) Φ is closed under finite unions and (ii) every closed subset of a member of Φ is in Φ). Recall that for $F \in \mathrm{Mod}(A_X)$, an element $s \in \Gamma(X; F)$ is in $\Gamma_\Phi(X; F)$ if and only if $\operatorname{supp} s$ is in Φ, i.e.

$$\Gamma_\Phi(X; F) = \varinjlim_{S \in \Phi} \Gamma(X; \Gamma_S F).$$

Later in the paper we shall use the right derived versions of many of the above formulas relating the various operations on A-sheaves. We will use these derived formulas freely and refer to reader to [24, Chapter II] for details. For instance, the cohomology with supports on Φ is defined by

$$H^*_\Phi(X; F) = R^*\Gamma_\Phi(X; F).$$

4.2. **O-minimal sheaves.** Let X be an object in Def and let A be a ring. The *o-minimal site* X_{def} on a definable space X is the category $\mathrm{Op}(X_{\mathrm{def}})$ whose objects are open definable subsets of X, the morphisms are the inclusions and the admissible covers $\mathrm{Cov}(U)$ of $U \in \mathrm{Op}(X_{\mathrm{def}})$ are covers by open definable subsets with finite subcoverings. We will denote by $\mathrm{Mod}(A_{X_{\mathrm{def}}})$ the category of sheaves of A-modules on X.

The tilde functor $\mathrm{Def} \to \widetilde{\mathrm{Def}}$ determines a morphism of sites

$$\nu_X : \widetilde{X} \to X_{\mathrm{def}}$$

given by the functor

$$\nu_X^t : \mathrm{Op}(X_{\mathrm{def}}) \to \mathrm{Op}(\widetilde{X}) : U \mapsto \widetilde{U}.$$

Theorem 4.1 ([13])**.** *The inverse image of $\nu_X : \widetilde{X} \to X_{\mathrm{def}}$ determines an isomorphism of categories*

$$\mathrm{Mod}(A_{X_{\mathrm{def}}}) \to \mathrm{Mod}(A_{\widetilde{X}}) : F \mapsto \widetilde{F},$$

where $\mathrm{Mod}(A_{\widetilde{X}})$ *is the category of A-sheaves on the topological space $\widetilde{X}$.*

The functors f_* and $\mathrm{Hom}_{A_{X_{\mathrm{def}}}}(\bullet, \bullet)$ commute with the tilde functor by definition. From this one can see that f^{-1} commutes by adjunction.

By Theorem 4.1 to develop sheaf theory in Def is equivalent to developing sheaf theory in $\widetilde{\mathrm{Def}}$. For instance, if X is an object of Def and if Φ is a family of definable supports on X (i.e. a collection of closed definable subsets of X such that: (i) Φ is closed under finite unions and (ii) every closed definable subset of a member of Φ is in Φ), then $\widetilde{\Phi}$, the collection of all closed subsets of tildes of members of Φ, is a family of supports on $\widetilde{X}$ and we set

$$H^*_\Phi(X; F) = H^*_{\widetilde{\Phi}}(\widetilde{X}; \widetilde{F}).$$

In the paper [15] we used this approach to develop the theory of Φ-supported sheaves, where Φ is definably normal, namely a family of definable supports supports such that: (1) each element of Φ is definably normal, (2) for each $S \in \Phi$ and each open definable neighborhood U of S there exists a closed definable neighborhood of S in U which is in Φ. Below we will use this theory and refer the reader to [15] for details.

Remark 4.2. Note that in [15] we assumed that A is a field, but this is only used there when dealing with the tensor product operation $\bullet \otimes_{A_X} G$ on A-sheaves (so that it is always exact). Here we will not require this operation.

Also it is often useful to use Theorem 4.1 to define new operations on o-minimal sheaves. For example, we can extend the usual definition of the extension by zero operation $i_{U!} : \mathrm{Mod}(U_{\mathrm{def}}) \to \mathrm{Mod}(X_{\mathrm{def}})$ on A-sheaves on a site where $U \in \mathrm{Op}(X_{\mathrm{def}})$, to the extension by zero operation $i_{Z!} : \mathrm{Mod}(Z_{\mathrm{def}}) \to \mathrm{Mod}(X_{\mathrm{def}})$ on A-sheaves on a site where Z is a definable locally closed subset of X, by setting

$$\widetilde{i_{Z!}F} = \widetilde{i}_{\widetilde{Z}!}\widetilde{F}.$$

It is useful to recall here the following general criteria:

Fact 4.3. *Let X an object of* Def *(resp. a definably normal object of* Def*) and let $\mathfrak{N}$ be a class of objects of* $\mathrm{Mod}(A_{X_{\mathrm{def}}})$*. Suppose that $\mathfrak{N}$ satisfies:*

(i) *for each exact sequence $0 \to F' \to F \to F'' \to 0$ with $F' \in \mathfrak{N}$ we have $F \in \mathfrak{N}$ if and only if $F'' \in \mathfrak{N}$;*
(ii) *$\mathfrak{N}$ is stable under filtrant $\varinjlim$;*
(iii) *$A_V \in \mathfrak{N}$ for any $V \in \mathrm{Op}(X_{\mathrm{def}})$ (resp. $A_U \in \mathfrak{N}$ for any $U \in \mathrm{Op}(X_{\mathrm{def}})$ such that $\overline{U}$ is affine).*

Then $\mathfrak{N} = \mathrm{Mod}(A_{X_{\mathrm{def}}})$.

This is the o-minimal analogue of the corresponding criteria in the semi-algebraic case ([9, Lemma 4.18]) and is obtained by using the isomorphism $\mathrm{Mod}(A_{X_{\mathrm{def}}}) \to \mathrm{Mod}(A_{\widetilde{X}})$ of Theorem 4.1 and applying the corresponding criteria in the topological case ([5, Chapter II, 15.10]) (point (ii) is a little bit stronger here) observing that constructible open subsets of $\widetilde{X}$ form a filtrant basis for the topology of $\widetilde{X}$.

On the other hand, suppose that X is a definably normal object of Def. Let $V \in \mathrm{Op}(X_{\mathrm{def}})$. Then by the shrinking lemma V has a finite cover $\{U_i\}_{i=1}^m$ consisting of open definable subsets of V such that each $\overline{U_i}$ is affine. So if $A_{U_i} \in \mathfrak{R}$, then by (ii) $A_V = \varinjlim_i A_{U_i} \in \mathfrak{R}$ and the result follows also.

4.3. **Invariance.** Here we prove our general criteria for invariance of o-minimal sheaf cohomology with definably compact supports and without supports.

Below we let $\mathbb{S}$ be a model of the first-order theory of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$).

Recall that given X an object of Def, there is a continuous surjective map $r : \widetilde{X(\mathbb{S})} \to \widetilde{X}$ defined as follows: for each $\alpha \in \widetilde{X(\mathbb{S})}$, $r(\alpha) = \{A : \ \alpha \in A(\mathbb{S})\}$. If $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, then the adjunction morphism $\mathrm{id} \to Rr_* \circ r^{-1}$ together with the isomorphisms $\mathrm{Mod}(A_{X_{\mathrm{def}}}) \to \mathrm{Mod}(A_{\widetilde{X}})$ and $\mathrm{Mod}(A_{X(\mathbb{S})_{\mathrm{def}}}) \to \mathrm{Mod}(A_{\widetilde{X(\mathbb{S})}})$ of Theorem 4.1 define a morphism

$$R\Gamma(X;F) \simeq R\Gamma(\widetilde{X};\widetilde{F}) \to R\Gamma(\widetilde{X};Rr_*r^{-1}\widetilde{F}) \simeq R\Gamma(X(\mathbb{S});F(\mathbb{S})). \tag{2}$$

where $F(\mathbb{S}) \in \mathrm{Mod}(A_{X(\mathbb{S})_{\mathrm{def}}})$ is the unique object such that $\widetilde{F(\mathbb{S})} = r^{-1}\widetilde{F}$, since $R\Gamma(\widetilde{X};Rr_*r^{-1}\widetilde{F}) \simeq R\Gamma(\widetilde{X(\mathbb{S})};r^{-1}\widetilde{F})$.

Below we shall use the criteria in Fact 4.3 to prove general invariance results. But first we make a couple of observations.

Remark 4.4. Let X be an object of Def and let $U \in \mathrm{Op}(X_{\mathrm{def}})$. The exact sequence $0 \to A_U \to A_{\overline{U}} \to A_{\overline{U}\setminus U} \to 0$ implies the following morphisms of distinguished triangles

$$\begin{array}{ccccccc}
R\Gamma(X;A_U) & \longrightarrow & R\Gamma(X;A_{\overline{U}}) & \longrightarrow & R\Gamma(X;A_{\overline{U}\setminus U}) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
R\Gamma(\overline{U};A_U) & \longrightarrow & R\Gamma(\overline{U};A_{\overline{U}}) & \longrightarrow & R\Gamma(\overline{U};A_{\overline{U}\setminus U}) & \longrightarrow &
\end{array}$$

where the vertical morphisms are determined by the adjunction $\mathrm{id} \to Ri_{\overline{U}*}i_{\overline{U}}^{-1}$ and $i_{\overline{U}} : \overline{U} \to X$ is the inclusion. If we set $Z = \overline{U}, \overline{U} \setminus U$, then we have

$$R\Gamma(\overline{U};A_Z) \simeq R\Gamma(X;A_Z) \simeq R\Gamma(Z;A_Z).$$

Therefore we have an isomorphism

$$R\Gamma(\overline{U};A_U) \simeq R\Gamma(X;A_U). \tag{3}$$

In the same way, working in $\mathrm{Def}(\mathbb{S})$ we obtain the isomorphism

$$R\Gamma(\overline{U}(\mathbb{S});A_{U(\mathbb{S})}) \simeq R\Gamma(X(\mathbb{S});A_{U(\mathbb{S})}). \tag{4}$$

Remark 4.5. Let X be an object of Def and let $U \in \mathrm{Op}(X_{\mathrm{def}})$. The exact sequence $0 \to A_U \to A_X \to A_{X\setminus U} \to 0$ implies the following morphisms of distinguished triangles

$$\begin{tikzcd} R\Gamma(X;A_U) \arrow[r] \arrow[d] & R\Gamma(X;A_X) \arrow[r] \arrow[d] & R\Gamma(X;A_{X\setminus U}) \arrow[r] \arrow[d] & {} \\ R\Gamma(X(\mathbb{S});A_{U(\mathbb{S})}) \arrow[r] & R\Gamma(X(\mathbb{S});A_{X(\mathbb{S})}) \arrow[r] & R\Gamma(X(\mathbb{S});A_{(X\setminus U)(\mathbb{S})}) \arrow[r] & {} \end{tikzcd}$$

where the vertical morphisms are given in (2).

If we set $Z = X \setminus U$, then we have

$$R\Gamma(X;A_Z) \simeq R\Gamma(Z;A_Z).$$

In the same way, working in $\mathrm{Def}(\mathbb{S})$, we have

$$R\Gamma(X(\mathbb{S});A_{Z(\mathbb{S})}) \simeq R\Gamma(Z(\mathbb{S});A_{Z(\mathbb{S})}).$$

Therefore, if $R\Gamma(Y;A_Y) \simeq R\Gamma(Y(\mathbb{S});A_{Y(\mathbb{S})})$ for $Y = X, X\setminus U$, then we have an isomorphism

$$R\Gamma(X;A_U) \simeq R\Gamma(X(\mathbb{S});A_{U(\mathbb{S})}). \tag{5}$$

We are ready to prove our first general invariance result:

Theorem 4.6. *Let* $\mathbf{C}$ *be a full subcategory of* Def *whose set of objects*

- *contains every closed definable subset of an object of* $\mathbf{C}$

and is such that:

(C1) *every object* X *of* $\mathbf{C}$ *is definably normal;*
(C2) *for every affine object* X *of* $\mathbf{C}$ *we have an isomorphism*

$$H^*(X;A_X) \simeq H^*(X(\mathbb{S});A_{X(\mathbb{S})}).$$

If X *is an object of* $\mathbf{C}$ *and* $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, *then we have an isomorphism*

$$H^*(X;F) \simeq H^*(X(\mathbb{S});F(\mathbb{S})).$$

Proof. Set $\mathfrak{S} = \{F \in \mathrm{Mod}(A_{X_{\mathrm{def}}}) : R\Gamma(X;F) \simeq R\Gamma(X(\mathbb{S});F(\mathbb{S}))\}$. We will obtain the result applying Fact 4.3 since we have (C1).

The family $\mathfrak{S}$ satisfies (i) and (ii) of Fact 4.3: the first is standard the second follows from the fact that sections commute with filtrant $\varinjlim$. So we are reduced to proving that the family $\mathfrak{S}$ satisfies (iii) of Fact 4.3. Now this follows by the isomorphisms (3) and (4) of Remark 4.4, (C2) and Remark 4.5 applied to $X = \overline{U}$. $\square$

We are ready to prove our criteria for invariance of o-minimal sheaf cohomology with definably compact supports:

Theorem 4.7. *Let* $\mathbf{B}$ *be a full subcategory of* Def *whose set of objects*

- *contains every closed definable subset of an object of* $\mathbf{B}$

and is such that:

(B1) *for every object* X *of* $\mathbf{B}$ *we have:*
  i. X *is definably normal;*
  ii. $X(\mathbb{S})$ *is* $\mathbb{S}$*-definably normal;*
(B2) *every object* X *in* $\mathbf{B}$ *has a definable completion in* $\mathbf{B}$*;*

(B3) *for every affine definably compact object X of $\mathbf{B}$ we have an isomorphism*
$$H^*(X; A_X) \simeq H^*(X(\mathbb{S}); A_{X(\mathbb{S})}).$$

If X is an object of $\mathbf{B}$ and $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, then we have an isomorphism
$$H^*_c(X; F) \simeq H^*_c(X(\mathbb{S}); F(\mathbb{S})).$$

Proof. By (B2) there is an open definable immersion $\iota : X \to P$ with P a definably compact object of $\mathbf{B}$. By Proposition 3.9, c is a definably normal family of supports on P. Therefore, by [15, Corollary 3.9] we have $H^*_c(X; F) \simeq H^*(P; \iota_! F)$. Similarly there is an open $\mathbb{S}$-definable immersion $\iota^{\mathbb{S}} : X(\mathbb{S}) \to P(\mathbb{S})$ with $P(\mathbb{S})$ an $\mathbb{S}$-definably compact object of $\mathbf{B}(\mathbb{S})$ (the image of $\mathbf{B}$ under the functor $\mathrm{Def} \to \mathrm{Def}(\mathbb{S})$). By Proposition 3.9 in $\mathbb{S}$, c is a normal and constructible family of supports on $P(\mathbb{S})$. Therefore, by [15, Corollary 3.9] we have $H^*_c(X(\mathbb{S}); F(\mathbb{S})) \simeq H^*(P(\mathbb{S}); (\iota^{\mathbb{S}})_! F(\mathbb{S}))$. We used here the invariance of: definable open immersion (Remark 2.10), definably normal ((B1)) and definably compact (Proposition 3.4).

Since $(\iota_! F)(\mathbb{S}) \simeq (\iota^{\mathbb{S}})_! F(\mathbb{S})$ (applying tilde we are in the case of diagram (1) on page 14) we obtain $H^*(P; \iota_! F) \simeq H^*(P(\mathbb{S}); (\iota^{\mathbb{S}})_! F(\mathbb{S}))$ by (B1), (B3) and Theorem 4.6 and therefore we have $H^*_c(X; F) \simeq H^*_c(X(\mathbb{S}); F(\mathbb{S}))$ as required. □

5. Applications

In this section we apply our general invariance criteria to obtain the invariance results stated in the Introduction. We also prove here the o-minimal analogues of Wilder's finiteness theorem mentioned in the Introduction.

5.1. Invariance in o-minimal expansions of ordered groups.

Here we assume that $\mathbb{M}$ is an o-minimal expansion of an ordered group. As before, we let $\mathbb{S}$ be a model of the first-order theory of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$).

From [10, Chapter 10, (1.8)] we have:

Fact 5.1. *In o-minimal expansions of real closed fields regular definable spaces are affine.*

However, in linear o-minimal expansions of ordered groups there are regular definable spaces with are not affine ([19]).

From [10, Chapter 6, (3.5)] we have:

Fact 5.2. *In o-minimal expansions of ordered groups, every affine definable space is definably normal.*

By Fact 5.2 and Proposition 2.20 we have:

Proposition 5.3. *If X and Y are definably normal definable spaces, then the following hold:*

1. *$X \times Y$ is definably normal;*
2. *X is completely definably normal;*
3. *$X(\mathbb{S})$ is $\mathbb{S}$-definably normal.*

From [1, Theorems 8.1 and 8.3] we have:

Fact 5.4. *For every closed and bounded definable subset $X \subseteq M^n$ we have an isomorphism*

$$H^*(X; A_X) \simeq H^*(X(\mathbb{S}); A_{X(\mathbb{S})}).$$

Proof of Theorem 1.1: Let X be a definably normal definable space with a definably normal definable completion P. We have to show that if F is a sheaf on the o-minimal site on X, then we have

$$H^*_c(X; F) \simeq H^*_c(X(\mathbb{S}); F(\mathbb{S})).$$

We obtain the result applying Theorem 4.7 taking $\mathbf{B}$ to the full subcategory of Def whose set of objects consists of: P, X, every closed definable subset of X and every closed definable subset of P. Then we have that: (B1) holds by assumption on X and P and by and Proposition 5.3; (B2) holds by assumption; (B3) holds by Fact 5.4. $\square$

From [18, Corollary 1.3] we have:

Fact 5.5. *If $\mathbb{M}$ is an o-minimal expansion of a real closed field, then for every definable subset $X \subseteq M^n$ we have an isomorphism*

$$H^*(X; A_X) \simeq H^*(X(\mathbb{S}); A_{X(\mathbb{S})}).$$

Corollary 5.6. *If $\mathbb{M}$ is an o-minimal expansion of a real closed field, then for every regular definable space X and every $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$ we have an isomorphism*

$$H^*(X; F) \simeq H^*(X(\mathbb{S}); F(\mathbb{S})).$$

Proof. We obtain the result applying Theorem 4.6 taking $\mathbf{C}$ to the full subcategory of Def whose set of objects consists of: X and every closed definable subset of X. Then we have that: (C1) holds by assumption on X and by Facts 5.1 and 5.2; (C2) holds by Fact 5.5.

$\square$

5.2. **Invariance in definably compact groups.** Here we assume that $\mathbb{M}$ is an arbitrary o-minimal structure and, as before, we let $\mathbb{S}$ be a model of the first-order theory of $\mathbb{M}$ (resp. an o-minimal expansion of $\mathbb{M}$).

Here the goal is to prove Theorem 1.4 in the Introduction. To proceed we require the following ([20, Definition 3.1]):

Definition 5.7. A *definable group-interval* $J = \langle (-b, b), 0, +, < \rangle$ is an open interval $(-b, b) \subseteq M$, with $-b < b$ in $M \cup \{-\infty, +\infty\}$, together with a binary partial continuous definable operation $+ : J^2 \to J$ and an element $0 \in J$, such that:

- $x + y = y + x$ (when defined), $(x + y) + z = x + (y + z)$ (when defined) and $x < y \Rightarrow x + z < y + z$ (when defined);
- for every $x \in J$ with $0 < x$, the set $\{y \in J : 0 < y \text{ and } x + y \text{ is defined}\}$ is an interval of the form $(0, r(x))$;
- for every $x \in J$ with $0 < x$, then $\lim_{z\to 0}(x+z) = x$ and $\lim_{z\to r(x)^-}(x+z) = b$;
- for every $x \in J$ there exists $z \in J$ such that $x + z = 0$.

The definable group-interval J is *unbounded* (resp. *bounded*) if the operation $+$ in J is total (resp. not total). The notion of a *definable homomorphism* between definable group-intervals is defined in the obvious way.

By the properties above, it follows that: (i) for each $x \in J$ there is a unique $z \in J$ such that $x + z = 0$, called the inverse of x and denoted by $-x$; (ii) for each $x \in J$ we have $-0 = 0$, $-(-x) = x$ and $0 < x$ if and only if $-x < 0$; (iii) the maps $J \to J : x \mapsto -x$ and $(-b, 0) \to (0, b) : x \mapsto -x$ are continuous definable bijections; (iv) for every $x \in J$ with $x < 0$, the set $\{y \in J : y < 0 \text{ and } x+y \text{ is defined}\}$ is an interval of the form $(-r(x), 0)$; (v) for every $x \in J$ with $x < 0$, then $\lim_{z\to 0}(x+z) = x$ and $\lim_{z\to -r(x)^+}(x + z) = -b$; (vi) for every $x \in J$ we have $x + 0 = x$ (both sides are defined and they are equal).

By the proof of [20, Lemma 3.4] we have:

Fact 5.8. *Let $J = \langle(-b, b), 0, +, -, <\rangle$ is a definable group-interval. Then there exists an injective, continuous definable homomorphism $\tau : J \to J$ given by $\tau(x) = \frac{x}{4}$ such that if $x, y \in \tau(J) = (-\frac{b}{4}, \frac{b}{4})$, then $x + y$, $x - y$ and $\frac{x}{2}$ are defined in J.*

From now on we fix a cartesian product $\mathbb{J}_m = \Pi_{i=1}^m J_i$ of definable group-intervals $J_i = \langle(-_ib_i, b_i),\, 0_i, +_i, -_i, <\rangle$.

We say that X is a $\mathbb{J}_m$*-bounded subset* if $X \subseteq \Pi_{i=1}^m[-_ic_i, c_i]$ for some $c_i > 0_i$ in J_i.

Remark 5.9. We will often identify a $\mathbb{J}_m$-bounded subset X with its image under the cartesian product of the injective homomorphisms given by Fact 5.8 and assume that $X \subseteq \Pi_{i=1}^m[-_ic_i, c_i]$ for some $0_i < c_i < \frac{b_i}{4}$ in J_i.

We say that X is a $\mathbb{J}_m$*-definable subset* if X is a definable set and $X \subseteq \Pi_{i=1}^m J_i$.

Let $l \in \{1, \dots, m-1\}$. For a $\mathbb{J}_l$-definable subset $X \subseteq \Pi_{i=1}^l J_i$, we set $L^l(X) = \{f : X \to J_{l+1} : f \text{ is definable and continuous}\}$ and $L^l_\infty(X) = L^l(X) \cup \{-_{l+1}b_{l+1}, b_{l+1}\}$, where we regard $-_{l+1}b_{l+1}$ and b_{l+1} as constant functions on X. If $f \in L^l(X)$, we denote by $\Gamma(f)$ the graph of f. If $f, g \in L^l_\infty(X)$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and set $(f, g)_X = \{(x, y) \in X \times J_{l+1} : f(x) < y < g(x)\}$. Then,

- a $\mathbb{J}_1$-*cell* is either a singleton subset of J_1, or an open interval with endpoints in $J_1 \cup \{-b, b_1\}$,
- a $\mathbb{J}_{l+1}$-*cell* is a set of the form $\Gamma(f)$, for some $f \in L^l(X)$, or $(f, g)_X$, for some $f, g \in L^l_\infty(X)$, $f < g$, where X is a $\mathbb{J}_l$-cell.

In either case, X is called *the domain* of the defined cell. The *dimension* of a $\mathbb{J}_m$-cell is defined as usual ([10, Chapter 3 (2.3) and Chapter 4 (1.1)]).

We refer the reader to [10, Chapter 3 (2.10)] for the definition of a *decomposition* of $\mathbb{J}_m$. A $\mathbb{J}_m$*-decomposition* is then a decomposition $\mathcal{C}$ of $\mathbb{J}_m$ such that each $B \in \mathcal{C}$ is a $\mathbb{J}_m$-cell. The following can be proved similarly to [10, Chapter 3 (2.11)].

Theorem 5.10 ($\mathbb{J}_m$-CDT)**.**

(1) *Given any $\mathbb{J}_m$-definable subsets $A_1, \dots, A_k$, there is a $\mathbb{J}_m$-decomposition $\mathcal{C}$ that partitions each A_i.*

(2) *Given any $f \in L^{m-1}(A)$, there is a $\mathbb{J}_m$-decomposition $\mathcal{C}$ that partitions A such that the restriction $f_{|B}$ to each $B \in \mathcal{C}$ with $B \subseteq A$ is continuous.*

Below we will need the following observations. If $\mathbb{J}_m$ is a cartesian product of bounded definable group-intervals, then there is an associated definable o-minimal structure $\widehat{\mathbb{J}}_m$ such that: (i) the domain of $\widehat{\mathbb{J}}_m$ is the definable set $\widehat{J} = (-_1 b_1, b_1) \cup \{-_2 b_2\} \cup (-_2 b_2, b_2) \cup \ldots \cup \{-_m b_m\} \cup (-_m b_m, b_m)$ with the obvious induced definable total order; (ii) the $\widehat{\mathbb{J}}_m$-definable subsets are the subsets $X \subseteq \widehat{J}^k$ such that X is a definable set.

Lemma 5.11. *The o-minimal structure $\widehat{\mathbb{J}}_m$ has $\widehat{\mathbb{J}}_m$-definable choice.*

Proof. Using Remark 5.9, this is obtained by suitably adapting the proof of [10, Chapter 6 (1.2)].

For $X \subseteq \widehat{J}$ a $\widehat{\mathbb{J}}_m$-definable and nonempty set, let x_0 be the least element of X if it exists, otherwise, let $(a, b) \subseteq X$ be the "left-most" interval given by $a = \inf X$ and $b = \sup\{x \in \widehat{J} : (a, x) \subseteq X\}$. Define $e(X)$ to be either x_0 if it exists or otherwise

$$e(X) = \begin{cases} -_2 b_2 & \text{if} \quad a = -_1 b_1, b = b_m, \\ b -_i |\frac{b}{2}|_i & \text{if} \quad a = -_1 b_1, b \in J_i, \\ a +_i |\frac{a}{2}|_i & \text{if} \quad a \in J_i, b = b_m, \\ b_i & \text{if} \quad a \in J_i, b \in J_l, i < l, \\ \frac{a +_i b}{2} & \text{if} \quad a, b \in J_i. \end{cases}$$

where for $c \in J_i$ we set as usual

$$|c|_i = \begin{cases} c & \text{if} \quad 0_i < c, \\ -_i c & \text{if} \quad c < 0_i. \end{cases}$$

For $X \subseteq \widehat{J}^n$ $(n > 1)$ a $\widehat{\mathbb{J}}_m$-definable and nonempty set, we define by induction on n, $e(X) = (e(\pi(X)), e(X_{e(\pi(X))}))$ where $\pi : \widehat{J}^n \to \widehat{J}^{n-1}$ is the projection onto the first $n-1$ coordinates and $X_a : \{z \in \widehat{J} : (a, z) \in X\}$ for every $a \in \pi(X)$.

If $\{S_a : a \in A\}$ is a $\widehat{\mathbb{J}}_m$-definable family of $\widehat{\mathbb{J}}_m$-definable and nonempty sets, then define a definable choice $f : A \to \bigcup_{a \in A} S_a$ by $f(a) = e(S_a)$ for every $a \in A$. $\square$

Remark 5.12. Let X be a $\mathbb{J}_m$-definable subset. Then X is a $\widehat{\mathbb{J}}_m$-definable set and a definable subset of X is relatively open if and only if it is a relatively open $\widehat{\mathbb{J}}_m$-definable subset of X. Therefore, we have that: (i) the o-minimal site of X in $\mathbb{M}$ is the same as the o-minimal site of X in $\widehat{\mathbb{J}}_m$; (ii) the o-minimal cohomology of X computed in $\mathbb{M}$ is the same as the o-minimal cohomology of X computed in $\widehat{\mathbb{J}}_m$.

Below we let L be a A-module.

As in the case of o-minimal expansions of ordered groups ([1, Corollary 3.3]) we have:

Lemma 5.13. *Let C be a $\mathbb{J}_m$-cell which is a $\mathbb{J}_m$-bounded subset. Then C is acyclic, i.e. $H^p(C; L_C) = 0$ for $p > 0$ and $H^0(C; L_C) = L$.*

Proof. This is obtained in exactly the same way as [1, Corollary 3.3]. Indeed, since C is a $\mathbb{J}_m$-bounded subset, by Remark 5.9, we can apply the group-interval

operations $x -_i y$, $x -_i y$ and $\frac{x}{2}$ in each coordinate of $\Pi_{i=1}^m J_i$ just like in the proof of [1, Corollary 3.3] obtaining:

Claim 5.14. *If I is a definably connected $\mathbb{J}_1$-bounded subset, then I is definably contractible to a point in J_1.*

Claim 5.15. *If C is a $\mathbb{J}_m$-cell which is a $\mathbb{J}_m$-bounded subset, then there is a definable deformation retract of C to a $\mathbb{J}_m$-cell which is a $\mathbb{J}_m$-bounded subset of strictly lower dimension.*

See [1, Lemmas 3.1 and 3.2].

By Claim 5.15 and induction on the dimension of C, C definably contractible to a point in $\Pi_{i=1}^m J_i$. Note also that by construction the domain of the definable deformation retraction of Claim 5.15 is a $\mathbb{J}_m$-definable subset. Therefore, by Lemma 5.11 and Remark 5.12, we have the homotopy axiom for o-minimal cohomology ([13]) for definable homotopies whose domains are $\mathbb{J}_m$-definable subsets. So $H^p(C; L_C)$ is the same as the o-minimal cohomology of a point and we apply the dimension axiom for o-minimal cohomology to conclude. $\square$

We also have the analogue of [1, Lemma 7.1]:

Lemma 5.16. *Let C be a $\mathbb{J}_m$-cell which is a $\mathbb{J}_m$-bounded subset and of dimension r. There is a definable family $\{C_{t_1,\dots,t_m} : 0_i < t_i < \frac{b_i}{4},\ i = 1,\dots,m\}$ of closed and $\mathbb{J}_m$-bounded subset $C_{t_1,\dots,t_m} \subset C$ such that:*

1. $C = \bigcup_{t_1,\dots,t_m} C_{t_1,\dots,t_m}$.
2. *If $0_i < t'_i < t_i$ for all $i = 1,\dots,m$, then $C_{t_1,\dots,t_m} \subset C_{t'_1,\dots,t'_m}$ and this inclusion induces an isomorphism*
$$H^p(C \backslash C_{t_1,\dots,t_m}; L_C) \simeq H^p(C \backslash C_{t'_1,\dots,t'_m}; L_C).$$
3. *The o-minimal cohomology of $C \backslash C_{t_1,\dots,t_m}$ is given by*
$$H^p(C \backslash C_{t_1,\dots,t_m}; L_C) = \begin{cases} L^{1+\chi_1(r)} & \text{if} \quad p \in \{0, r-1\} \\ \\ 0 & \text{if} \quad p \notin \{0, r-1\}. \end{cases}$$
where $\chi_1 : \mathbb{Z} \to \{0,1\}$ is the characteristic function of the subset $\{1\}$.

Proof. By Remark 5.9, we assume that $C \subseteq \Pi_{i=1}^m [-_i c_i, c_i]$ for some $0_i < c_i < \frac{b_i}{4}$ in J_i and the group-interval operations $x -_i y$, $x -_i y$ and $\frac{x}{2}$ are all defined in each coordinate of $\Pi_{i=1}^m J_i$.

We define the definable family $\{C_{t_1,\dots,t_m} : 0_i < t_i < \frac{b_i}{4},\ i = 1,\dots,m\}$ by induction on $l \in \{1,\dots,m-1\}$ in the following way.

1. If $l = 1$ and C is a singleton in J_1, we define $C_{t_1} = C$.
2. If $l = 1$ and $C = (d, e) \subseteq J_1$, then $C_{t_1} = [d +_1 \gamma^1_{t_1}, e -_1 \gamma^1_{t_1}]$ where $\gamma^1_{t_1} = \min\{|\frac{d -_1 e}{2}|_1, t_1\}$, (in this way C_{t_1} is non empty).
3. If $l > 1$ and $C = \Gamma(f)$, where $f \in L^l(B)$ is a continuous definable map and B is $\mathbb{J}_l$-cell which is a $\mathbb{J}_l$-bounded subset. By induction $B_{t_1,\dots,t_l}$ is defined. We put $C_{t_1,\dots,t_l,t_{l+1}} = \Gamma(f_{|B_{t_1,\dots,t_l}})$.
4. If $l > 1$ and $C = (f, g)_B$, where $f, g \in L^l(B)$ are continuous definable maps, B is $\mathbb{J}_l$-cell which is a $\mathbb{J}_l$-bounded subset and $f < g$. By induction $B_{t_1,\dots,t_l}$

is defined. We put $C_{t_1,\dots,t_l,t_{l+1}} = [f +_{l+1} \gamma^{l+1}_{t_{l+1}}, g -_{l+1} \gamma^{l+1}_{t_{l+1}}]_{B_{t_1,\dots,t_l}}$, where $\gamma^{l+1}_{t_{l+1}} := \min(|\frac{f -_{l+1} g}{2}|_{l+1}, t_{l+1})$.

We observe that from this construction we obtain:

Claim 5.17. *For $t_1, \dots, t_m$ as above there is a covering $\mathcal{U}_C = \{U_i : i \in I\}$ of $C \backslash C_{t_1,\dots,t_m}$ by relatively open $\mathbb{J}_m$-bounded subset such that:*

1. *The index set I is the family of the closed faces of an r-dimensional cube. (So $|I| = 2r$).*
2. *If $E \subset I$, then $U_E := \bigcap_{i \in E} U_i$ is either empty or a $\mathbb{J}_m$-cell. (So in particular $H^p(U_E; L_C) = 0$ for $p > 0$ and, if $U_E \neq \emptyset, H^0(U_E; L_C) = L$.)*
3. *For $E \subset I$, $U_E \neq \emptyset$ iff the faces of the cubes belonging to E have a non-empty intersections.*

So the nerve of $\mathcal{U}_C$ is isomorphic to the nerve of a covering of an r-cube by its closed faces.

Proof. To prove that there is a covering satisfying the properties above, we define $\mathcal{U}_C$ by induction on $l \in \{1, \dots, m-1\}$. We distinguish four cases according to definition of the $C_{t_1,\dots,t_m}$.

1. If $l = 1$ and C is a singleton in J_1, then $\mathcal{U}_C$ is the covering consisting of one open set (given by the whole space C).
2. If $l = 1$ and $C = (d, e) \subseteq J_1$, then $C \backslash C_{t_1}$ is the union of the two open subsets $(d, d +_1 \gamma^1_{t_1})$ and $(e -_1 \gamma^1_{t_1}, e)$, and we define $\mathcal{U}_C$ as the covering consisting of these two sets.
3. If $l > 1$ and $C = \Gamma(f)$, where $f \in L^l(B)$ is a continuous definable map and B is $\mathbb{J}_l$-cell which is a $\mathbb{J}_l$-bounded subset. By definition $C_{t_1,\dots,t_l,t_{l+1}} = \Gamma(f_{|B_{t_1,\dots,t_l}})$. By induction we have a covering $\mathcal{V}_B$ of $B \backslash B_{t_1,\dots,t_l}$ with the stated properties, and we define $\mathcal{U}_C$ to be a covering of $C \backslash C_{t_1,\dots,t_l,t_{l+1}}$ induced by the natural homeomorphism between the graph of f and its domain.
4. If $l > 1$ and $C = (f, g)_B$ where $f, g \in L^l(B)$ are continuous definable maps, B is $\mathbb{J}_l$-cell which is a $\mathbb{J}_l$-bounded subset and $f < g$. By definition $C_{t_1,\dots,t_l,t_{l+1}} = [f +_{l+1} \gamma^{l+1}_{t_{l+1}}, g -_{l+1} \gamma^{l+1}_{t_{l+1}}]_{B_{t_1,\dots,t_l}}$. By induction we have that $B \backslash B_{t_1,\dots,t_l}$ has a covering $\mathcal{V}_B = \{V_j : j \in J\}$ with the stated properties, where J is the set of closed faces of the cube $[0,1]^{r-1}$. Define a covering $\mathcal{U}_C = \{U_i : i \in I\}$ of $C \backslash C_{t_1,\dots,t_l,t_{l+1}}$ as follows. As index set I we take the closed faces of the cube $[0,1]^r$. Thus $|I| = |J| + 2$, with the two extra faces corresponding to the"top" and "bottom" face of $[0,1]^r$. We associate to the top face the open set $(g -_{l+1} \gamma^{l+1}_{t_{l+1}}, g)_{B_{t_1,\dots,t_l}} \subset C \backslash C_{t_1,\dots,t_l,t_{l+1}}$ and the bottom face the open set $(f, f +_{l+1} \gamma^{l+1}_{t_{l+1}})_{B_{t_1,\dots,t_l}} \subset C \backslash C_{t_1,\dots,t_l,t_{l+1}}$. The other open sets of the covering are the preimages of the sets V_j under the restriction of the projection $\Pi^{l+1}_{i=1} J_i \to \Pi^l_{i=1} J_i$. This defines a covering of $C \backslash C_{t_1,\dots,t_l,t_{l+1}}$ with the stated properties. $\square$

Property (1) of the lemma is clear. By (the proof of) Claim 5.17 there are open covers $\mathcal{U}'_C$ of $C \backslash C_{t'_1,\dots,t'_m}$ and $\mathcal{U}_C$ of $C \backslash C_{t_1,\dots,t_m}$ satisfying the assumptions of [1, Lemma 5.5]. Hence property (2) of the lemma holds. Finally, if $r > 1$, then property (3) follows from Claim 5.17 and [1, Corollary 5.2]. On the other hand, if $r = 1$, then $C \setminus C_{t_1,\dots,t_m}$ is by construction a disjoint union $D \sqcup E$ of two $\mathbb{J}_m$-cells which are

$\mathbb{J}_m$-bounded subsets and of dimension $r = 1$. Therefore, in this case, the result follows from Lemma 5.13, since $H^*(C\backslash C_{t_1,\dots,t_m}; L_C) \simeq H^*(D; L_D) \oplus H^*(E; L_E)$. $\square$

From Lemma 5.16 and computations in o-minimal cohomology we obtain just like in [1, Lemma 7.2 and Corollary 7.3]:

Lemma 5.18. *Let X be a $\mathbb{J}_m$-definable, $\mathbb{J}_m$-bounded subset and $C \subseteq X$ a $\mathbb{J}_m$-cell of maximal dimension. Then for every $t_1, \dots, t_m$ and $t'_1, \dots, t'_m$ with $t'_i < t_i$ for all $i = 1, \dots, m$ as above we have isomorphisms induced by inclusions:*
(1) $H^*(X\backslash C_{t_1,\dots,t_m}; L_X) \simeq H^*(X\backslash C_{t'_1,\dots,t'_m}; L_X)$;
(2) $H^*(X\backslash C_{t_1,\dots,t_m}; L_X) \simeq H^*(X\backslash C; L_X)$ *assuming also that X is closed.*

Proof. (1) Since C is an open definable subset of X, by excision we have

$$H^*_{C_{t'_1,\dots,t'_m}}(X\backslash C_{t_1,\dots,t_m}; L_X) \simeq H^*_{C_{t'_1,\dots,t'_m}}(C\backslash C_{t_1,\dots,t_m}; L_X).$$

On the other hand, by Lemma 5.16 and the long exact cohomology sequence of the pair $(C\backslash C_{t_1,\dots,t_m}, C\backslash C_{t'_1,\dots,t'_m})$ we have

$$H^*_{C_{t'_1,\dots,t'_m}}(C\backslash C_{t_1,\dots,t_m}; L_X) = 0.$$

Therefore, $H^*_{C_{t'_1,\dots,t'_m}}(X\backslash C_{t_1,\dots,t_m}; L_X) = 0$ and the result follows by the long exact cohomology sequence of the pair $(X\backslash C_{t_1,\dots,t_m}, X\backslash C_{t'_1,\dots,t'_m})$.

(2) Follows from (1) and [1, Lemma 6.7]. $\square$

Remark 5.19. Let X be a $\mathbb{J}_m$-definable, $\mathbb{J}_m$-bounded subset and $C \subseteq X$ a $\mathbb{J}_m$-cell. Assume that $C \subseteq \Pi_{i=1}^m[-_ic_i, c_i]$ for some $0_i < c_i < \frac{b_i}{4}$ in J_i. Then there is a point $p_C \in C$ such that for all $t_1, \dots, t_m$ as above, if $c_i < t_i$ for all $i = 1, \dots, m$, then $C_{t_1,\dots,t_m} = \{p_C\}$. In particular, we have

$$H^*(\overline{C} \setminus \{p_C\}; L_X) \simeq H^*(\overline{C} \setminus C; L_X)$$

even if $\overline{C}$ is in general non-acyclic ([1, Theorem 4.1]).

From Lemma 5.18 and computations in o-minimal cohomology we obtain just like in [1, Theorems 8.1]:

Lemma 5.20. *If X is a closed, $\mathbb{J}_m$-definable, $\mathbb{J}_m$-bounded subset, then we have an isomorphism*

$$H^*(X; A_X) \simeq H^*(X(\mathbb{S}); A_{X(\mathbb{S})}).$$

Proof. Take a $\mathbb{J}_m$-cell decomposition of X and let $C \subseteq X$ a $\mathbb{J}_m$-cell of maximal dimension. Take $\overline{t} = t_1, \dots, t_m$ as before and write $C_{\overline{t}}$ instead of $C_{t_1,\dots,t_m}$ for short. Then we have a commutative diagram

$$\begin{tikzcd}
H^{i-1}(C\backslash C_{\overline{t}}) \arrow[r] \arrow[d] & H^i(X) \arrow[r] \arrow[d] & H^i(X\backslash C_{\overline{t}}) \oplus H^i(C) \arrow[r] \arrow[d] & H^i(C\backslash C_{\overline{t}}) \arrow[r] \arrow[d] & {} \\
H^{i-1}((C\backslash C_{\overline{t}})(\mathbb{S})) \arrow[r] & H^i(X(\mathbb{S})) \arrow[r] & H^i((X\backslash C_{\overline{t}})(\mathbb{S})) \oplus H^i(C(\mathbb{S})) \arrow[r] & H^i((C\backslash C_{\overline{t}})(\mathbb{S})) \arrow[r] & {}
\end{tikzcd}$$

(where we omitted the coefficients) given by the Mayer-Vietoris sequences for X and $X(\mathbb{S})$. Now apply the five lemma, Lemmas 5.13, 5.16 and 5.18 together with induction on the number of cells to conclude that all vertical arrows are isomorphisms. $\square$

We can now go back to the proof of Theorem 1.4 in the Introduction. Below we assume the reader familiarity with the basic theory of definable groups ([25] and [27]).

From [17, Corollary 2.3] we have:

Fact 5.21. *If G is a definably compact definable group, then G is definably normal.*

Corollary 5.22. *Let G be a definably compact definable group. Then $G(\mathbb{S})$ is $\mathbb{S}$-definably compact and $\mathbb{S}$-definably normal.*

Proof. From Fact 5.21 and Proposition 3.4, $G(\mathbb{S})$ is $\mathbb{S}$-definably compact. Hence, by Fact 5.21 in $\mathbb{S}$ we have that $G(\mathbb{S})$ is $\mathbb{S}$-definably normal. $\square$

Recall the notation introduced in [17]. In a definable group G we have that: (i) the identity element $e \in G$ has a uniformly definable family of open definable neighborhoods O^{ϵ} with $\epsilon = \langle \epsilon_1^-, \epsilon_1^+, \ldots, \epsilon_n^-, \epsilon_n^+ \rangle$ where $n = \dim G$; (ii) if $d \in G$, then $O^{\epsilon}(d) = dO^{\epsilon}$ (resp. $\overline{O}^{\epsilon}(d) = \overline{dO^{\epsilon}}$) is a uniformly definable family of open (resp. closed) neighborhoods of d in G; if $D \subseteq G$ is a definable set, then $O^{\epsilon}(D) = \cup\{O^{\epsilon}(d) : d \in D\}$ is an uniformly definable family of open definable neighborhood of D in G and we set $\overline{O}^{\epsilon}(D) = \cup\{\overline{O}^{\epsilon}(d) : d \in D\}$. Moreover, if $\delta = \langle \delta_1^-, \delta_1^n, \ldots, \delta_n^-, \delta_n^+ \rangle$ is such that $\epsilon_i^- < \delta_i^- < \delta_i^+ < \epsilon_i^+$ for all $i = 1, \ldots, n$, then $\overline{O}^{\delta}(D) \subseteq O^{\epsilon}(D)$.

We have ([17, Proposition 2.2]):

Fact 5.23. *Let G be a definable group. If K is a definably compact, definable subset of G and U is an open definable neighborhood of K in G, then there a definably compact, definable neighborhood of K in U of the form $\overline{O}^{\delta}(K)$.*

Corollary 5.24. *If G is a definably compact definable group, then G is completely definably normal.*

Proof. Let U be an open definable subset of G. We have to show that U is definably normal. Let A be a (relatively) closed definable subset of U and let V be a (relatively) open definable neighborhood of A in U. We have to find a (relatively) closed definable neighborhood D of A in V.

The set $K = \overline{A} \cap (G \setminus V)$ a closed (and hence definably compact) definable subset of G. By Fact 5.23, there exists a definably compact neighborhood of K in G of the form $\overline{O}^{\epsilon}(K)$ for $\epsilon = \langle \epsilon_1^-, \epsilon_1^+, \ldots, \epsilon_n^-, \epsilon_n^+ \rangle$ where $n = \dim G$. Choose $\delta = \langle \delta_1^-, \delta_1^n, \ldots, \delta_n^-, \delta_n^+ \rangle$ such that $\epsilon_i^- < \delta_i^- < \delta_i^+ < \epsilon_i^+$ for all $i = 1, \ldots, n$. Let $B = A \cap (G \setminus O^{\delta}(K))$. Then $B \subseteq V$ is closed in G (hence definably compact) and so V is an open definable neighborhood of B in G. By Fact 5.23,, there exists a definably compact neighborhood of B in V of the form $\overline{O}^{\eta}(B)$. Let $D = \overline{O}^{\eta}(B) \cup (\overline{O}^{\epsilon}(K) \cap V)$. Then: (i) $A \subseteq D$ (if $x \in A$, then $x \in B \subseteq \overline{O}^{\eta}(B)$ or $x \in O^{\delta}(K) \subseteq \overline{O}^{\epsilon}(K)$); (ii) $D \subseteq V$; (iii) D is a (relatively) closed definable neighborhood of A in V. $\square$

We will require the following result ([20, Theorem 3]):

Fact 5.25. *If G is a definable group, then there is a definable injection $G \to \Pi_{i=1}^m J_i$, where each $J_i \subseteq M$ is a definable group-interval.*

The following is also useful:

Corollary 5.26. *Let G be a definably compact definable group. Then there is a cartesian product $\mathbb{J}_m = \Pi_{i=1}^m J_i$ of bounded definable group-intervals and there is a definably compact definable group H which is a $\mathbb{J}_m$-definable $\mathbb{J}_m$-bounded subset. Such that:*

1. *G is definably isomorphic, hence definably homeomorphic, to H.*
2. *The definable manifold structure on H is such that for each definable chart (U_l, ϕ_l), $\phi_l(U_l)$ is a $\mathbb{J}_m$-definable $\mathbb{J}_m$-bounded subset.*
3. *H is definable in the definable o-minimal structure $\widehat{\mathbb{J}}_m$ with $\widehat{\mathbb{J}}_m$-definable choice.*

Proof. Suppose that $J_i = \langle (-_i b_i, b_i), 0_i, +_i, -_i, < \rangle$. By Fact 5.25, G is definably isomorphic to a definable group $H \subseteq \Pi_{i=1}^m J_i$ (which is therefore a $\mathbb{J}_m$-definable subset). By [27] definable isomorphisms of definable groups are definable homeomorphisms when each definable group is equipped with its definable manifold structure. This, by Lemma 5.11, it remains to prove (2).

Without loss of generality, assume that $G = H \subseteq \Pi_{i=1}^m J_i$. By the construction of the definable manifold structure of G ([27]), G is a definable space whose definable charts (U_l, ϕ_l) are such that each U_l is a $\mathbb{J}_m$-definable subset. In fact, each U_l is a $\mathbb{J}_m$-cell in $G \subseteq \Pi_{i=1}^m J_i$ of dimension $n = \dim G$ or U_l is a translate in G of a $\mathbb{J}_m$-cell in $G \subseteq \Pi_{i=1}^m J_i$ of dimension n. In the first case, ϕ_l is the restriction of a projection from $\Pi_{i=1}^m J_i$ onto some $n < m$ coordinates. In the second case ϕ_l is the composition of a translation in G and the restriction of a projection as above. For the fact that the restriction of a projection as above is a definable homeomorphism compare with [10, Chapter 3, (2.7)].

Consider the open definable subsets $V_l \subseteq U_l$ (for each l) given by [17, Corollary 2.4] such that $\overline{V_l} \subseteq U_l$ and $G = \bigcup_l V_l$. It is enough to show that for every U_l which is a $\mathbb{J}_m$-cell in $G \subseteq \Pi_{i=1}^m J_i$, any definably compact definable subset C of G such that $C \subseteq U_l \subseteq \Pi_{i=1}^m J_i$ is $\mathbb{J}_m$-bounded.

Fix l such that U_l is a $\mathbb{J}_m$-cell in $G \subseteq \Pi_{i=1}^m J_i$ and suppose that C is a definably compact definable subset of G such that $C \subseteq U_l$ and C is not $\mathbb{J}_m$-bounded. Then there is a j such that the projection of C onto the j-coordinate is not bounded. Since G has definable choice ([14, Theorem 7.2]) one of the following holds: (i) there is a definable map $\alpha : (e, b_j) \subseteq (-_j b_j, b_j) \longrightarrow U_l \subseteq G$ such that $\operatorname{im}\alpha \subseteq C$ and for each $t \in (e, b_j)$, we have $\alpha_j(t) > t$ where $\alpha_j(t)$ is the j-coordinate of $\alpha(t)$; (ii) there is a definable map $\alpha : (-_j b_j, d) \subseteq (-_j b_j, b_j) \longrightarrow U_i \subseteq G$ such that $\operatorname{im}\alpha \subseteq C$ and for each $t \in (-_j b_j, d)$, we have $\alpha_j(t) < t$ where $\alpha_j(t)$ is the j-coordinate of $\alpha(t)$. We assume (i) holds. For (ii) the proof is similar. By o-minimality we may assume that α is continuos with respect to the topology of G. Since C is definably compact, the limit $\lim_{t \to b_i} \alpha(t)$, with respect to the topology induced by G on C, exists in C. Let a be this limit.

By the observation in the first paragraph, the topology induced by G on U_l is the same as the topology induced by $\Pi_{i=1}^m J_i$ on U_l. Let $B = \Pi_{i=1}^m (-_i c_i, c_i) \subseteq \Pi_{i=1}^m J_i$ for some $c_i > 0_i$ in J_i, such that B contains a. Then $B \cap U_l$ is an open definable neighborhood of a in $U_l \subseteq G$ in the topology of G. Thus there is a $t_0 \in (e, b_j)$ such that $\operatorname{im}\alpha_{|(t_0, b_j)} \subseteq B \cap U_l \subseteq B$. But this is absurd since $\operatorname{im}\alpha_{j|(t_0,b_j)}$ is not $\mathbb{J}_m$-bounded. $\square$

Proof of Theorem 1.4: Let G be a definably compact definable group. We have to show that if F is a sheaf on the o-minimal site on G, then we have

$$H^*(G;F) \simeq H^*(G(\mathbb{S});F(\mathbb{S})).$$

We obtain the result applying Theorem 4.6 taking $\mathbf{C}$ to the full subcategory of Def whose set of objects consists of: G and every closed definable subset of G. Then we have that: (C1) holds by Fact 5.21; (C2) holds by Corollary 5.26 and Lemma 5.20. □

5.3. **O-minimal Wilder's finiteness theorem.** Here we prove the o-minimal analogue of Wilder's finiteness theorem ([23, III.10]).

Recall that a definable space X is *definably locally compact* if for every definably compact definable subset K of X and every open definable neighborhood U of K in X, there is a definably compact definable neighborhood C of K in U.

Lemma 5.27. *Let A be a noetherian ring. Let X be a definably locally compact, definably normal definable space. Let $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$. Suppose that for every affine definably compact subset B of X, $H^i(B;F)$ is finitely generated for each $i \in \mathbb{Z}$. Then for every pair (Z,K) of definably compact definable subsets of X such that $K \subseteq \mathring{Z}$, the restriction map*

$$H^i(Z;F) \to H^i(K;F)$$

has finitely generated image for each $i \in \mathbb{Z}$.

Proof. The proof in by induction on i. For $i=0$, we have that Z (resp. K) has finitely many definably connected components, say m_Z (resp. m_K) of them. Since $K \subseteq Z$, we have $m_Z \leq m_K$. On the other hand, $H^0(K;F) \simeq F(K)^{m_K}$ and $H^0(Z;F) \simeq F(Z)^{m_Z}$, and the result follows.

Assume the result holds in degrees $< i$. Let $\mathcal{A}$ be the collection of all definably compact definable subsets A of X for which there exists a definably compact subset C of X with $A \subseteq \mathring{C} \subseteq C \subseteq \mathring{Z}$ such that the restriction map $H^i(Z;F) \to H^i(C;F)$ has finitely generated image.

Claim 5.28. *The collection $\mathcal{A}$ has the following properties:*

1. *If A is a definably compact definable subset of X such that $A \subseteq \mathring{Z}$ and A is a subset of definably chart of X, then $A \in \mathcal{A}$.*
2. *If $A \in \mathcal{A}$ and $R \subseteq A$ is a definably compact subset of A, then $R \in \mathcal{A}$.*
3. *If $A \in \mathcal{A}$ and $R \in \mathcal{A}$, then $A \cup R \in \mathcal{A}$.*

We obtain (1) by assumption and the fact that X is definably locally compact. (2) Is clear. For (3), suppose that $A \in \mathcal{A}$ and $R \in \mathcal{A}$. Since X is definably locally compact, find definably compact subsets B and C such that $A \subseteq \mathring{B} \subseteq B \subseteq \mathring{C} \subseteq C \subseteq \mathring{Z}$ such that the restriction map $H^i(Z;F) \to H^i(C;F)$ has finitely generated image. Similarly, find definably compact subsets S and T such that $R \subseteq \mathring{S} \subseteq S \subseteq \mathring{T} \subseteq T \subseteq \mathring{Z}$ such that the restriction map $H^i(Z;F) \to H^i(T;F)$ has finitely generated image. Consider the following commutative diagram constructed

from the Mayer-Vietoris sequences

$$\begin{tikzcd}
 & H^i(Z;F) \arrow[r] \arrow[d] & H^i(Z;F) \oplus H^i(Z;F) \arrow[d] \\
H^{i-1}(C\cap T;F) \arrow[r] \arrow[d] & H^i(C\cup T;F) \arrow[r] \arrow[d] & H^i(C;F) \oplus H^i(T;F) \\
H^{i-1}(B\cap S;F) \arrow[r] & H^i(B\cup S;F). &
\end{tikzcd}$$

Note that: (i) the middle horizontal sequence of the diagram is exact; the first down arrow on the bottom square of the diagram has finitely generated image (by the induction hypothesis); the second down arrow on the top square of the diagram has finitely generated image (by the hypothesis of (3)). By the purely algebraic result [23, III. Lemma 10.3], we conclude that the restriction map $H^i(Z;F) \to H^i(B\cup S;F)$ is an isomorphism and hence $A \cup R \in \mathcal{A}$.

Now let (Z,K) be a pair of definably compact definable subsets of X such that $K \subseteq \mathring{Z}$. Since X is definably normal, by the shrinking lemma, we have $K = K_1 \cup \cdots \cup K_r$ where each K_i is a definably compact subset of X such that $K_i \subseteq \mathring{Z}$ and K_i is a subset of definable chart of X. We conclude the proof by induction on r. If $r = 0$, then by Claim 5.28 (1), the restriction map $H^i(Z;F) \to H^i(K;F)$ has finitely generated image for each $i \in \mathbb{Z}$. The inductive step follows from Claim 5.28 (3). $\square$

Corollary 5.29. *Let A be a noetherian ring. Let X be a definably compact, definably normal definable space. Let $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$. Suppose that for every affine closed definable subset B of X, $H^i(B;F)$ is finitely generated for each $i \in \mathbb{Z}$. Then $H^i(X;F)$ is finitely generated for each $i \in \mathbb{Z}$.*

As before, we fix a cartesian product $\mathbb{J}_m = \Pi_{i=1}^m J_i$ of definable group-intervals $J_i = \langle(-_i b_i, b_i), 0_i, +_i, -_i, <\rangle$.

Also we obtain just like in [1, Theorem 7.4]:

Lemma 5.30. *Let A be a noetherian ring and we let L be a finitely generated A-module. Let X be a closed $\mathbb{J}_m$-definable, $\mathbb{J}_m$-bounded subset. Then, for each p, $H^p(X;L_X)$ is finitely generated.*

Proof. Take a $\mathbb{J}_m$-cell decomposition of X and let $C \subseteq X$ a $\mathbb{J}_m$-cell of maximal dimension $r > 0$ (if $r = 0$ then X is a point and the result follows since L is finitely generated). Take $\overline{t} = t_1, \ldots, t_m$ as before and write $C_{\overline{t}}$ instead of $C_{t_1,\ldots,t_m}$ for short. Then we have a Mayer-Vietoris sequence

$$H^{i-1}(C\backslash C_{\overline{t}}) \longrightarrow H^i(X) \longrightarrow H^i(X\backslash C_{\overline{t}}) \oplus H^i(C) \longrightarrow H^i(C\backslash C_{\overline{t}}) \longrightarrow$$

(where we omitted the coefficients) for X. Now apply Lemmas 5.13, 5.16 and 5.18 we get from this the exact sequences

$$0 \longrightarrow H^0(X) \longrightarrow H^0(X\backslash C) \oplus H^0(C) \longrightarrow L^{1+\chi_1(r)}$$

$$L^{1+\chi_1(r)} \longrightarrow H^r(X) \longrightarrow H^r(X\backslash C) \oplus H^r(C) \longrightarrow 0$$

$$L^{\chi_1(r-1)} \longrightarrow H^{r-1}(X) \longrightarrow H^{r-1}(X\backslash C) \oplus H^{r-1}(C) \longrightarrow L^{1+\chi_1(r)}$$

and

$$0 \longrightarrow H^i(X) \longrightarrow H^i(X\backslash C) \oplus H^i(C) \longrightarrow 0$$

for $i \neq r-1, r$ (where as before we omitted the coefficients, $\chi_1 : \mathbb{Z} \to \{0,1\}$ is the characteristic function of the subset $\{1\}$ and we set $L^0 = 0$). By induction on the number of cells, Lemma 5.13 and the fact that L and so $L^{1+\chi_1(r)}$ is noetherian (being a finitely generated A-module with A noetherian) the result follows. □

Corollary 5.31. *Let A be a noetherian ring and we let L be a finitely generated A-module. Let X be a definably compact, definably normal definable space whose definable charts (X_i, ϕ_i) are such that each $\phi_i(X_i)$ is a $\mathbb{J}_m$-definable, $\mathbb{J}_m$-bounded subset. Then, for each p, $H^p(X; L_X)$ is finitely generated.*

Proof. By Lemma 5.30, the assumption of Corollary 5.29 holds. □

Proof of Theorem 1.5: Suppose that $\mathbb{M}$ is an o-minimal expansion of an ordered group. Suppose that L is a finitely generated module over a noetherian ring. Let X be a definably compact, definably normal definable space whose definable charts (X_i, ϕ_i) are such that each $\phi_i(X_i)$ is a bounded definable set. By Corollary 5.31 with each J_i a sub group-interval of the underlying ordered group of $\mathbb{M}$, we obtain that, for each p, $H^p(X; L_X)$ is finitely generated. □

Proof of Theorem 1.6: Suppose that $\mathbb{M}$ is an arbitrary o-minimal structure. Suppose that L is a finitely generated module over a noetherian ring. Let G be a definably compact definable group. By Corollary 5.31 and Corollary 5.26, we obtain that, for each p, $H^p(G; L_G)$ is finitely generated. □

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